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$W^{2,k}_{Cl_n}$-BEST APPROXIMATION OF A γ-REGULAR FUNCTION

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Abstract. In this paper, we construct γ-regular Cl_n-minimal function systems in $W^{2,k}_{\Gamma}(\Omega, Cl_n) \cap \ker D_\gamma(\Omega, Cl_n)$, the generalized Bergman space of Cl_n-valued functions in the Sobolev space $W^{2,k}_{\Gamma}(\Omega)$ which are used in the best way to approximate null solutions of the in-homogeneous Dirac operator.

1. Introduction

Interpolations, approximations and integral transforms are methods used to solve problems of linear and non-linear partial differential equations. To this end, techniques of Clifford analysis play a very crucial role. The works in [1], [2], [6], [7], [9] and [10] are some to mention. In [1] and [2] the authors study elliptic boundary value problems over bounded and Liapunov domains using Quaternionic and Clifford analysis. In [5], [6] and [7] we work on elliptic boundary value problems over domains which are Lipschitz and unbounded for the first order Dirac operator and over more smooth domains for the higher order iterates of the Dirac operator. In [9] the

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author generalizes an existing function theory of the in-homogeneous Dirac operator $D_\lambda := D - \lambda$ which is studied by Z. Xu in [13], [14] to $D\gamma := D - \gamma$ where,

$$\gamma = \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j}\Gamma, \quad \text{for } \Gamma \in C^1(\Omega \to \mathbb{R})$$

and λ is some Clifford number. In this case, γ is called the *gradient potential* of Γ. In [6], [7], we construct a family of left regular functions called Cl_n-complete systems which are used to approximate null solutions of the Dirac operator and in [4], we do for the in-homogeneous Dirac operator. The theme of this paper is to construct an optimal family of γ-regular functions called Cl_n-minimal family, from the Cl_n-complete systems constructed in [4] which will give best approximations for left γ-regular functions in some Sobolev spaces, which is not the case in the Cl_n-complete systems. But first, we develop the necessary function theory for the in-homogeneous Dirac operator (2.3). The Cauchy kernel that we use here for our work is given by (see in [9])

$$\Psi^\Gamma(x-y) := \frac{\overline{(x-y)}}{\omega_n \|x-y\|^n} e^{-(\Gamma(x)-\Gamma(y))} \tag{1.1}$$

2. Preliminaries and function theoretic results

Let $\{e_j : j = 1, \ldots, n\}$ be an orthonormal basis of $\mathbb{R}^n$. Consider the 2^n-dimensional Clifford algebra Cl_n generated from $\mathbb{R}^n$ equipped with a negative inner product. Then we have the anti-commutation relationship $e_i e_j + e_j e_i = -2\delta_{ij} e_0$, $i, j = 1, 2, \ldots, n$, where δ_{ij} is the Kronecker delta symbol and e_0 is the identity element of Cl_n. The set $\{e_A\}_{A \subseteq \{1,\ldots,n\}}$, with $e_A = e_{i_1} \ldots e_{i_k}$, $e_{\{i\}} = e_i$, $i = 1, \ldots, n$ and $e_\varnothing = e_0$, are the basis of Cl_n. Thus each element of the algebra is represented in the form: $a = \sum_A a_A e_A$, where, a_A's are real numbers and then every element $x = (x_1, \ldots, x_n)$ of $\mathbb{R}^n$ is identified with the element $x = \sum_{k=1}^n x_k e_k$ of the Clifford algebra. In this way we have an embedding: $\mathbb{R}^n \hookrightarrow Cl_n$ of the vector space $\mathbb{R}^n$ into the algebra Cl_n. We also define the Clifford conjugate of $a = \sum_A a_A e_A \in Cl_n$ denoted by $\overline{a}$ as $\overline{a} = \sum_A a_A \overline{e}_A$, where $\overline{e}_A = \overline{e}_{i_k} \ldots \overline{e}_{i_1}$, $\overline{e}_j = -e_j, \overline{e}_0 = e_0$, $j = 1, \ldots, n$.

Definition 2.1. For an element $a = \sum_A a_A e_A \in Cl_n$, we define its Clifford norm by

$$\|a\| = \left(\sum_A a_A^2\right)^{1/2}.$$

An important point of Clifford analysis is that each non-zero element $x \in \mathbb{R}^n$ has an inverse

$$x^{-1} = \frac{\overline{x}}{\|x\|^2}.$$

This is Kelvin inversion up to a sign.

In what follows, $\Omega \subseteq \mathbb{R}^n$ is a bounded C^2-domain. A function defined in Ω with values in Cl_n has a representation: $f(x) = \sum_A e_A f_A(x)$, $x \in \Omega$, $f_A(x) \in \mathbb{R}$.

Such a function f is continuous, differentiable, integrable, measurable, etc. over Ω, if each component function f_A is respectively continuous, differentiable, integrable, measurable, etc. over Ω. Thus the usual function spaces denoted by $C^\alpha(\Omega, Cl_n)$, $L^p(\Omega, Cl_n)$ and $W^{p,k}(\Omega, Cl_n)$ for $k = 0, 1, \dots$ and $1 < p < \infty$, are defined as follows: $f \in C^\alpha(\Omega, Cl_n)$ if $f_A \in C^\alpha(\Omega, \mathbb{R})$, and $f \in W^{p,k}(\Omega, Cl_n)$ if $f_A \in W^{p,k}(\Omega, \mathbb{R})$.

Note here that $W^{p,0}(\Omega, Cl_n) = L^p(\Omega, Cl_n)$, $1 < p < \infty$.

For $p = 2$, the Lebesgue space $L^2(\Omega, Cl_n)$ becomes a Hilbert space with a Clifford-valued inner product given by

$$\langle f, g \rangle := \int_\Omega \overline{f(x)} g(x) d\Omega. \tag{2.1}$$

Let us introduce the Dirac operator by

$$D = \sum_{k=1}^{n} e_k \frac{\partial}{\partial x_k}. \tag{2.2}$$

This operator is a hypercomplex analogue of the well known complex Cauchy-Riemann operator $\partial_z = \partial_x + i\partial_y$. For the Dirac operator D, we have that $\Delta = \overline{D} D = D \overline{D}$, where Δ is the Laplacian and

$$\overline{D} = \sum_{k=1}^{n} \overline{e}_k \frac{\partial}{\partial x_k}$$

is the conjugate of the Dirac operator D.

Definition 2.2 ([9])**.** Let $\Gamma \colon \Omega \to \mathbb{R}$ be a C^1-function and let $\gamma(x) = \sum_{j=1}^{n} e_j \gamma_j(x)$. Then γ is called the gradient potential of Γ if and only if

$$\gamma(x) = \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j} \Gamma\left(x\right).$$

Then we introduce the in-homogeneous Dirac-operator with gradient potential γ by:

$$D_\gamma := \sum_{j=1}^{n} e_j \left(\frac{\partial}{\partial x_j} - \gamma_j \right) \tag{2.3}$$

where D is the Dirac-operator (2.2) and γ is the above defined gradient potential.

Definition 2.3. A function $h \in C^1(\Omega \to Cl_n)$ is said to be left γ-regular with respect to the potential Γ if $D_\gamma h(x) = 0$, $\forall x \in \Omega$, and right γ-regular with respect to Γ if $h(x)D_\gamma = 0$.

Let us denote by

$$M_{\gamma,l}(\Omega, Cl_n) = \{f \in C^1(\Omega, Cl_n) \colon D_\gamma f(x) = 0\}$$

and

$$M_l(\Omega, Cl_n) = \{f \in C^1(\Omega, Cl_n) \colon Df = 0\}.$$

The following lemma is an isomorphism between spaces of functions which are left-monogenic and left γ-regular over Ω.

Lemma 2.1 ([9])**.** *The modules $M_l(\Omega, Cl_n)$ and $M_{\gamma,l}(\Omega, Cl_n)$ are canonically isomorphic.*

Proof. Clearly the map $f \mapsto f(x)e^{-\Gamma(x)} \colon M_l(\Omega, Cl_n) \mapsto M_{\gamma,l}(\Omega, Cl_n)$ is an isomorphism. $\square$

Corollary 2.2. *The fundamental solution to the in-homogeneous Dirac-operator D_γ is given by*

$$\Psi^\Gamma(x-y) = \frac{\overline{(x-y)}}{\omega_n \|x-y\|^n} e^{-(\Gamma(x)-\Gamma(y))}, \quad x \neq y$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

Using this fundamental solution, we define the following convolution integral operators. But first let us define the necessary function spaces:

$$C^{0,\alpha}_\Gamma(\Omega, Cl_n) := \left\{f \colon \Omega \to Cl_n \colon f(x)\, e^{\Gamma(x)} \in C^{0,\alpha}(\Omega, Cl_n)\right\},$$
$$W^{p,k}_\Gamma(\Omega, Cl_n) := \left\{f : \Omega \to Cl_n \colon f(x)\, e^{\Gamma(x)} \in W^{p,k}(\Omega, Cl_n)\right\}$$

for $k = 0, 1, 2, \ldots$ and $p > 1$. Here when $k = 0$, we have weighted Lebesgue spaces $L^p_\Gamma(\Omega, Cl_n)$.

Definition 2.4. Let $f \in C^{0,\alpha}_\Gamma(\Omega) \cap C_\Gamma(\overline{\Omega})$, $0 < \alpha < 1$. Then define, the following Cl_n-valued convolution operators:

$$\zeta_{\Omega,\Gamma} f(x) := -\int_\Omega \Psi^\Gamma(x-y) f(y) d\Omega_y, \quad x \in \mathbb{R}^n \tag{2.4}$$

$$\zeta_{\Sigma,\Gamma} f(x) := \int_\Sigma \Psi^\Gamma(x-y)\nu(y) f(y) d\Sigma_y, \quad x \notin \Sigma \tag{2.5}$$

and

$$S_{\Sigma,\Gamma}f(x) := 2\int_{\Sigma} \Psi^{\Gamma}(x-y)\nu(y)f(y)d\Sigma_y, \quad x \in \Sigma \tag{2.6}$$

where $\nu(y)$ is the outer unit normal vector to the boundary Σ at the point y. Note here that the operator (2.6) is a singular integral operator and is understood in terms of the Cauchy principal value ($p.v.$).

Lemma 2.3 ([4]). *The operator $\zeta_{\Omega,\Gamma}$ is continuous from $W^{p,k}_{\Gamma}(\Omega, Cl_n)$ in to $W^{p,k+1}_{\Gamma}(\Omega, Cl_n)$ for $1 < p < \infty$, $k = 0, 1, \dots$.*

The in-homogeneous Dirac operator (2.3) has also a right inverse given by the integral operator defined in (2.4). But before we prove this, we have the following,

Proposition 2.4 ([4]). *Let $f \in C^{0,1}_{\Gamma}(\Omega) \cap C_{\Gamma}(\overline{\Omega})$. Then*

$$D\zeta_{\Omega,\Gamma}f = \gamma\zeta_{\Omega,\Gamma}f + f, \quad x \in \Omega$$

where D is the homogeneous Dirac operator given in (2.2).

Proof. For each $i = 1, \dots, n$ and for each $x \in \Omega$, we have

$$\begin{aligned}\frac{\partial}{\partial x_i}\zeta_{\Omega,\Gamma}f(x) &= \frac{\partial}{\partial x_i}\int_{\Omega}\Psi^{\Gamma}(x-y)f(y)d\Omega_y\\ &= \int_{\Omega}\left(\frac{e_i - n(x_i-y_i)\dfrac{(x-y)}{\|x-y\|^2}}{\|x-y\|^n} + \gamma_i(x)\frac{(x-y)}{\|x-y\|^n}\right)e^{-(\Gamma(x)-\Gamma(y))}f(y)d\Omega_y\\ &\quad - \frac{e_i}{n}f(x).\end{aligned}$$

Then

$$\begin{aligned}D\zeta_{\Omega,\Gamma}f(x) &= \int_{\Omega}\left(\frac{-n+n+\gamma(x)}{\|x-y\|^n}\right)e^{-(\Gamma(x)-\Gamma(y))}f(y)d\Omega_y + f(x)\\ &= \gamma(x)\int_{\Omega}\Psi^{\Gamma}(x-y)f(y)d\Omega_y + f(x).\end{aligned}$$

□

Corollary 2.5 ([4]). *Let f be a Cl_n-valued function which satisfies the condition of Proposition 2.4 above. Then*

$$D_{\gamma}\int_{\Omega}\Psi^{\Gamma}(x-y)f(y)d\Omega_y = f(x), \quad x \in \Omega$$

and

$$D_\gamma \int_\Omega \Psi^\Gamma(x-y)f(y)d\Omega_y = 0, \quad x \in \overline{\Omega}^c.$$

Corollary 2.6. *Let* $f \in W_\Gamma^{p,k}(\Omega)$, $1 < p < \infty$, $k = 0, 1, \ldots$. *Then*

$$D_\gamma \int_\Omega \Psi^\Gamma(x-y)f(y)d\Omega_y = f(x), \quad x \in \Omega$$

and

$$D_\gamma \int_\Omega \Psi^\Gamma(x-y)f(y)d\Omega_y = 0, \quad x \notin \overline{\Omega}.$$

The main relationships between the function f, the differential operator (2.3) and the convolution integrals (2.4), (2.5) and the nature of the singular integral transform (2.6) near the surface Σ are given in the following theorem.

Theorem 2.7. *Let* $f \in C_\Gamma^{0,1}(\Omega, Cl_n) \cap C_\Gamma(\overline{\Omega}, Cl_n)$. *Then*

$$f_{|\Omega} = \zeta_{\Sigma,\Gamma} f + \zeta_{\Omega,\Gamma} D_\gamma f. \tag{2.7}$$

If $f \in M_{\gamma,l}(\Omega, Cl_{0,n})$, then we have the Cauchy integral formula

$$f(x) = (\zeta_{\Sigma,\Gamma} f)(x) = \int_\Sigma \Psi^\Gamma(x-y)\nu(y)f(y)d\Sigma_y, \quad x \in \Omega.$$

Note that $\zeta_{\Sigma,\Gamma} f_{|\Omega} = 0$ if and only if $f_{|\Omega} = 0$. Thus γ-regular functions are determined from their traces on the boundary Σ of Ω.

Corollary 2.8 ([4])**.** *Let* $f \in W_\Gamma^{p,k}(\Omega, Cl_n)$, *for* $k = 0, 1, \ldots$ *and* $1 < p < \infty$. *Then*

$$\int_\Sigma \Psi^\Gamma(x-y)\nu(y)f(y)d\Sigma + \int_\Omega \Psi^\Gamma(x-y)D_\gamma f(y)d\Omega_y = f(x), \quad x \in \Omega. \tag{2.8}$$

Since also, $\zeta_{\Sigma,\Gamma} + \zeta_{\Omega,\Gamma} D_\gamma = I_\Omega$ on Ω, where I_Ω is the identity operator, we see that the operator

$$\zeta_{\Sigma,\Gamma} \colon W_\Gamma^{p,k-1/p}(\Sigma, Cl_n) \to W_\Gamma^{p,k}(\Omega, Cl_n) \cap \ker D_\gamma(\Omega)$$

is continuous. This is indeed, if $f \in W_\Gamma^{p,k-1/p}(\Sigma, Cl_n)$, then there exists $g \in W_\Gamma^{p,k}(\Omega, Cl_n)$ such that $\mathrm{tr}_\Sigma\, g = f$. Then from (2.8) and from the continuity of $\zeta_{\Omega,\Gamma}$, we have $D_\gamma \zeta_{\Sigma,\Gamma} f = 0$. That is $\zeta_{\Sigma,\Gamma} f \in M_{\gamma,l}(\Omega, Cl_n)$. That also means $(f - \zeta_{\Omega,\Gamma} D_\gamma f)_{|\Omega} \in M_{\gamma,l}(\Omega, Cl_n)$.

Again since the trace operator $\mathrm{tr}_{\partial\Omega}\colon W^{p,k}_{\Gamma}(\Omega) \to W^{p,k-1/p}_{\Gamma}(\Sigma)$ is continuous [12] and from the continuity of $\zeta_{\Sigma,\Gamma}$, we have that the singular integral operator $S_{\Sigma,\Gamma}\colon L^p_{\Gamma}(\Sigma) \to L^p_{\Gamma}(\Sigma)$ is continuous. For details see [1] and [2].

Proposition 2.9 ([4]). *Let $f \in C^{0,\alpha}_{\Gamma}(\Sigma, Cl_n)$, $0 < \alpha < 1$, and let Σ_{τ_Q} be a subset of Σ consisting of points with tangent spaces in Σ. For each $y \in \Sigma_{\tau_Q}$, let $\eta_y := y + t\nu(y)\colon (0,1] \to \mathbb{R}^n \setminus \Sigma$ where, $\nu(y)$ is the outer unit normal vector to Σ at point y. Then*

$$\lim_{t\downarrow 0} \int_{\Sigma} \Psi^{\Gamma}(x - \eta_y(t))\nu(x)f(x)d\Sigma - p.v. \int_{\Sigma} \Psi^{\Gamma}(x-y)\nu(x)f(x)d\Sigma = \frac{1}{2}f(y)$$

for each $y \in \Sigma_{\tau_Q}$.

And if $\rho_y := y - t\nu(y)\colon (0,1] \to \mathbb{R}^n \setminus \Sigma$, then

$$\lim_{t\downarrow 0} \int_{\Sigma} \Psi^{\Gamma}(x - \rho_y(t))\nu(x)f(x)d\Sigma - p.v. \int_{\Sigma} \Psi^{\Gamma}(x-y)\nu(x)f(x)d\Sigma = -\frac{1}{2}f(y)$$

for each $y \in \Sigma_{\tau_Q}$.

Here and in the next proposition, *p.v.* refers to the Cauchy principal value.

Proposition 2.10 ([4]). *Let $f \in W^{p,k}_{\Gamma}(\Sigma, Cl_n)$, for $1 < p$, $k = 0, \ldots,$ and let Σ_{τ_Q}, η_y, ρ_y be defined as above. Then*

$$\begin{aligned}&\lim_{t\downarrow 0} \int_{\Sigma} \Psi^{\Gamma}(x - y - t\nu(y))\nu(x)f(x)d\Sigma - p.v. \int_{\Sigma} \Psi^{\Gamma}(x-y)\nu(x)f(x)d\Sigma \\ &= \frac{1}{2}f(y)\end{aligned}$$

for $y \in \Sigma_{\tau_Q}$ and

$$\begin{aligned}&\lim_{t\downarrow 0} \int_{\Sigma} \Psi^{\Gamma}(x - y + t\nu(y))\nu(x)f(x)d\Sigma - p.v. \int_{\Sigma} \Psi^{\Gamma}(x-y)\nu(x)f(x)d\Sigma \\ &= -\frac{1}{2}f(y)\end{aligned}$$

for $y \in \Sigma_{\tau_Q}$.

Theorem 2.11 (Luzin's). *Let $f \in C^{0,1}_{\Gamma}(\Omega) \cap C_{\Gamma,0}(\overline{\Omega})$ and $D_\gamma f(x) = 0$ in Ω. Further, let $\Pi \subseteq \Sigma$ be a $(n-1)$-dimensional submanifold and $f(x) = 0$ on Π. Then $f(x) = 0$ in $\overline{\Omega}$.*

Observe that, if $f \in C_{\Gamma,0}(\Omega)$ and $D_\gamma f = 0$ on Π, then $f \equiv 0$ in Ω.

Proposition 2.12 ([4]). *The Lebesgue space $L^2_\Gamma(\Omega, Cl_n)$ has an orthogonal decomposition*

$$L^2_\Gamma(\Omega, Cl_n) = B^2_\gamma(\Omega, Cl_n) \oplus \overline{D}_\gamma(W^{2,1}_{\Gamma,0}(\Omega, Cl_n)) \tag{2.9}$$

with respect to the inner product given by equation (2.1).

$B^2_\gamma(\Omega, Cl_n)$ is the Bergman space of square integrable, left γ-regular Cl_n-valued functions over Ω, and $W^{2,1}_{\Gamma,0}(\Omega, Cl_n)$ is the space of functions in the Sobolev space $W^{2,1}_\Gamma(\Omega, Cl_n)$ whose traces vanish over the boundary Σ of Ω, or equivalently, the completion of $C^\infty_{\Gamma,0}(\Omega, Cl_n)$ in the Space $W^{2,1}_\Gamma(\Omega, Cl_n)$, and $L^2_\Gamma(\Omega, Cl_n)$ is the space of functions $f\colon \Omega \to Cl_n$ such that $f(x)e^{\Gamma(x)} \in L^2(\Omega, Cl_n)$. The decomposition gives orthogonal projection operators as well:

$$P\colon L^2_\Gamma(\Omega, Cl_n) \mapsto B^2_\gamma(\Omega, Cl_n) \text{ and } Q\colon L^2_\Gamma(\Omega, Cl_n) \mapsto \overline{D}_\gamma(W^{2,1}_{\Gamma,0}(\Omega, Cl_n))$$

such that for $f \in L^2_\Gamma(\Omega, Cl_n)$, $f = Pf + Qf$, with $PQ = QP = 0$ and $P^2 = P$, $Q^2 = Q$.

3. Cl_n-minimal function systems

In this part of the paper, we construct Cl_n-minimal family of functions in $B^2_\gamma(\Omega, Cl_n)$ which are more refined than the ones obtained in [4]. Similar and analogous results in quaternionic analysis are also obtained by K. Gürlebeck and W. Spröessig in [1], [2]. These functions are useful for approximating solutions of elliptic boundary value problems in the best way. For this purpose we choose dense points of some outer surface and define a family of functions from the fundamental solution (1.1) of the in-homogeneous Dirac operator (2.3) by making the selected points as the singular points of the fundamental solution. We then refine these functions more by an orthogonalization process. We begin with results of classical analysis.

Definition 3.1. Let the couple $(X, \|.\|)_{Cl_n}$ be a normed right-vector space X over the Clifford algebra Cl_n. A system of points $\{x_m\}_m \subseteq X$ is called Cl_n-complete in X if and only if

$$\forall \varepsilon > 0,\ \forall x \in X,\ \exists \lambda_i \in Cl_n \quad (i = 1, \dots, n_0)$$

such that

$$\left\| x - \sum_{i=1}^{n_0} x_i \lambda_i \right\|_X < \varepsilon.$$

Definition 3.2. A system of points $\{x_m\}_m \subseteq X$ is called closed in X if every bounded Cl_n-valued right linear functional Λ that vanishes on the points vanishes on the whole space X.

Lemma 3.1. *The system of points $\{x_m\}_m \subseteq X$ is closed if and only if it is Cl_n-complete in X.*

Proposition 3.2 ([4]). *Let Ω and Ω_l be bounded C^2-domains in $\mathbb{R}^n$ such that $\Omega_l \supseteq \overline{\Omega}$, $\Sigma := \partial\Omega$ and $\Sigma_l := \partial\Omega_l$. Let also $\{x_m\}_m$ be a dense subset of Σ_l. Then the family of functions $\Im := \left\{\Psi^\Gamma_m\right\}_m$ where, for each $m \in N$,*

$$\Psi^\Gamma_m(x) := \frac{\overline{(x - x_m)}}{\omega_n\|x - x_m\|^n} e^{-(\Gamma(x)-\Gamma(x_m))},$$

is Cl_n-complete system in $B^2_\gamma(\Omega, Cl_n)$.

Definition 3.3. A family of functions $\{f_i\}_i$ in $(X, \|\cdot\|)_{Cl_n}$, a normed right vector space of Cl_n-valued functions on X, is called Cl_n-minimal, if $\forall j$, $f_j \notin X \setminus \overline{[\operatorname{span}_{Cl_n}\{f_k\}_{k\neq j}]}$.

Then the Cl_n-complete function system $\{\Psi^\Gamma_m\}_m$ which we construct from a dense subset $\{x_m\}_m$ of an outer surface $\Sigma \subset \overline{\Omega}^c = \mathbb{R}^n \setminus \overline{\Omega}$ is not Cl_n-minimal in $B^2_\gamma(\Omega, Cl_n)$, since an other dense family of functions can be obtained from $\{x_m\}_m \setminus \{\text{finitely many of } x_m\text{'s}\}$

Definition 3.4. A finite family of functions $\{f_j : j = 1, \ldots, m\} \subset (X, \|\cdot\|)_{Cl_n}$ is called Cl_n-unisolvent with respect to $\{y_j : j = 1, \ldots, m\} \subset \Omega$, if the algebraic expression $\Lambda(x) = f_1(x)\lambda_1 + f_2(x)\lambda_2 + \ldots + f_m(x)\lambda_m$ vanishes at most at $n-1$ points of the set $\{y_j : j = 1, \ldots, m\}$, $\forall\lambda = (\lambda_j \in Cl_n)_{j=1,\ldots,n}$ with $\|\lambda\|^2 > 0$.

Proposition 3.3. *Let Ω, Ω_0 be two bounded C^2-domains with $\Omega_0 \supset \overline{\Omega}$ and $\operatorname{dist}(\partial\Omega, \partial\Omega_0) \geq \delta > 0$ for some δ. Let also $\{y_i\}_i$ be a dense subset of $\Sigma = \partial\Omega$ and set $x_i = y_i + t\nu(y_i)$ so that $x_i \in \partial\Omega_0$, where $\nu(y_i)$ is a unit pointing outer normal to $\partial\Omega$ at y_i. Then for some $\alpha > 0$, the system $\{\Psi^\Gamma_i : i = 1, \ldots, m\}$ is Cl_n-unisolvent with respect to the points $\{y_i : i = 1, \ldots, m\}$ for some t, $0 < t < \alpha$.*

Proof. Here we need to show that, if the algebraic equation $\Lambda(x) = \Psi^\Gamma_1\lambda_1 + \ldots + \Psi^\Gamma_m\lambda_m$ vanishes at each point y_i, for $i = 1, \ldots, m$, then each of the

Clifford numbers $\lambda_i = 0$, for $i = 1, \ldots, m$. Therefore consider the system of equations obtained from evaluating Λ at each point y_i:

$$\begin{aligned}\Lambda(y_1) =&\Psi_1^\Gamma(y_1)\lambda_1 + \Psi_2^\Gamma(y_1)\lambda_2 + \ldots + \Psi_m^\Gamma(y_1)\lambda_m = 0\\ \Lambda(y_2) =&\Psi_1^\Gamma(y_2)\lambda_1 + \Psi_2^\Gamma(y_2)\lambda_2 + \ldots + \Psi_m^\Gamma(y_2)\lambda_m = 0\\ &\vdots\\ \Lambda(y_m) =&\Psi_1^\Gamma(y_m)\lambda_1 + \Psi_2^\Gamma(y_m)\lambda_2 + \ldots + \Psi_m^\Gamma(y_m)\lambda_m = 0.\end{aligned}$$

Let us investigate the coefficients. First, the ones which are along the diagonal:

$$\begin{aligned}\|\Psi_k^\Gamma(y_k)\| &= \left\| \frac{\overline{(y_k - x_k)}}{\omega_n \|y_k - x_k\|^n} e^{-(\Gamma(y_k)-\Gamma(x_k))} \right\|\\ &= \left\| \frac{\overline{(y_k - (y_k + t\nu(y_k)))}}{\omega_n \|y_k - (y_k + t\nu(y_k))\|^n} e^{-(\Gamma(y_k)-\Gamma(y_k+t\nu(y_k)))} \right\|\\ &\geq \frac{\zeta}{\omega_n t^{n-1}} \to \infty \quad \text{as } t \to 0^+\end{aligned}$$

where ζ is some positive constant.

And the ones which are off the diagonal:

$$\begin{aligned}\|\Psi_k^\Gamma(y_j)\| &= \left\| \frac{\overline{(y_k - x_j)}}{\omega_n \|y_k - x_j\|^n} e^{-(\Gamma(y_k)-\Gamma(x_j))} \right\|\\ &= \left\| \frac{\overline{(y_k - (y_j + t\nu(y_j)))}}{\omega_n \|y_k - (y_j + t\nu(y_j))\|^n} e^{-(\Gamma(y_k)-\Gamma(y_j+t\nu(y_j)))} \right\|\\ &\leq \frac{\beta}{\|y_k - y_j - t\nu(y_j)\|^{n-1}} \to \frac{\beta}{\|y_k - y_j\|^{n-1}} < +\infty, \text{ as } t \to 0^+\end{aligned}$$

for some $\beta > 0$.

Therefore,

$$\|\Psi_k^\Gamma(y_k)\| \geq \sum_{\substack{j=1\\(j\neq k)}}^m \|\Psi_k^\Gamma(y_j)\| \quad \text{as } t \downarrow 0^+.$$

Which implies that $\exists \alpha > 0 \ni$ for $0 < t < \alpha$, we have

$$\|\Psi_k^\Gamma(y_k)\| \geq \sum_{\substack{j=1\\(j\neq k)}}^m \|\Psi_k^\Gamma(y_j)\|.$$

That is the coefficient matrix $\left(\Psi_k^\Gamma(y_j)\right)_{k,j}^m$ of the above homogeneous system $\Psi\lambda = 0$ is diagonally dominant. Therefore, the system has a unique solution for $\lambda = (\lambda_i : i = 1, m)$ and that is $\lambda = 0$. The above arguments show that the diagonal elements dominate the matrix of coefficients of the above system of equations and therefore the system has a unique solution for $\lambda_1, \lambda_2, \ldots, \lambda_m$. Hence $\lambda_1 = \lambda_2 = \ldots = \lambda_m = 0$. Which proves the result. $\square$

Remark 3.1. Even though the solvability of such a system is guaranteed in our case, for some other types of systems, there has to be a way to determine the optimal interval of existence for α.

Let $\{y_i\}_i$ and $\{x_i\}_i$ be as in Proposition 3.3. We construct a family of functions $\{\phi_i\}_i$ from $\{\Psi_i^\Gamma\}_i$ as follows:

for each $m \in N$, and $k = 1, \ldots, m$, $\phi_k = \Psi_k^\Gamma - \sum_{i=1}^{k-1} \phi_i \gamma_{i,k}$, with $\phi_1 := \Psi_1^\Gamma$, and for $i < k$, $\gamma_{i,k} \in Cl_n$, are determined in such a way that:

$$\begin{gathered}
\phi_1(y_1)\gamma_{12} = \Psi_2^\Gamma(y_1), \\
\phi_1(y_2)\gamma_{13} + \phi_2(y_2)\gamma_{23} = \Psi_3^\Gamma(y_2), \\
\vdots \\
\phi_1(y_i)\gamma_{1,i+1} + \phi_2(y_i)\gamma_{2,i+1} + \ldots + \phi_i(y_i)\gamma_{i.i+1} = \Psi_{i+1}^\Gamma(y_i), \\
\vdots \\
\phi_1(y_k)\gamma_{1,k+1} + \phi_2(y_k)\gamma_{2,k+2} + \ldots + \phi_k(y_k)\gamma_{k,k+1} = \Psi_{k+1}^\Gamma(y_k).
\end{gathered}$$

Then we have: $\phi_i(y_i) \neq 0$, $\forall i \in N$, and $\phi_i(y_j) = 0$, for $j < i$.

Proposition 3.4. *Let $\{y_i\}_i$ and $\{x_i\}_i$ be as in Proposition 3.3. Then the family of functions $\{\phi_i\}_i$ constructed above is Cl_n-minimal in $B_\gamma^2(\Omega, Cl_n)$.*

Proof. First, we note that from the construction of the functions ϕ_i's, $\phi_i(y_i) \neq 0$, $\forall i \leq m$, $m \in N$. Also $\{\phi_i\}_i$ is Cl_n-complete. This is indeed, if $f \in B_\gamma^2(\Omega, Cl_n)$ is such that $\int_\Omega \overline{\phi}_i(x) f(x) d\Omega_x = 0$, $i = 1, \ldots$, then $\int_\Omega \overline{\Psi}_i^\Gamma(x) f(x) d\Omega_x = 0$, for $i = 1, \ldots$. This implies that $f \equiv 0$, since $\{\Psi_i^\Gamma\}_i$ is Cl_n-complete. Thus, we proved the completeness of $\{\phi_i\}_i$. To prove that $\{\phi_i\}_i$ is Cl_n-minimal, we assume the contrary. That is, there exists $\phi_{k_0} \in B_\gamma^2(\Omega, Cl_n) \setminus \overline{[\underset{Cl_n}{\text{span}}\{\phi_i\}_{i \neq k_0}]}$. Then we get:

$$\phi_{k_0} = \lim_{m \to \infty} (\phi_1 \gamma_{1,m} + \phi_2 \gamma_{2,m} + \ldots + \phi_m \gamma_{m,m}) \text{ in } B_\gamma^2(\Omega, Cl_n).$$

From Harnack's convergence theorem, we have

$$\phi_{k_0}(x) = \lim_{m\to\infty}(\phi_1(x)\gamma_{1,m} + \phi_2(x)\gamma_{2,m} + \ldots + \widehat{\phi}_{k_0}(x)\gamma_{k_0,m} + \ldots \\ + \phi_m(x)\gamma_{m,m}), \quad x \in \Omega$$

where $\widehat{\phi}_{k_0}$ is omitted. From the construction, we have $\phi_{k_0}(y_j) = 0$, for $j < k_0$ and hence

$$0 = \phi_{k_0}(y_j) = \lim_{m\to\infty} \sum_{i=1,\, i\neq k_0}^{m} \phi_i(y_j)\gamma_{i,m} = \lim_{m\to\infty} \sum_{i=1}^{j} \phi_j(y_j)\gamma_{j,m}.$$

This implies $\lim_{m\to\infty} \gamma_{j,m} = 0$. This in turn implies:

$$\phi_{k_0}(y_{k_0}) = \lim_{m\to\infty} \sum_{i=1,\, i\neq k_0}^{m} \phi_i(y_{k_0})\gamma_{i,m} = \lim_{m\to\infty} \sum_{i=1}^{k_0-1} \phi_i(y_{k_0})\gamma_{i,m} = 0.$$

This is a contradiction to our construction that $\phi_{k_0}(y_{k_0}) \neq 0$. This proves that $\{\phi_i\}_i$ is Cl_n-minimal in $B^2_\gamma(\Omega, Cl_n)$. □

Proposition 3.5. *Let*

$$B_{(m)} := \underset{Cl_n}{\mathrm{span}}\{\phi_k \colon k = 1, \ldots, m\}$$

and $f \in B^2_\gamma(\Omega, Cl_n)$ *such that* $f = \phi_1\lambda_1 + \ldots + \phi_m\lambda_m$. *If* $p_m(f) = \sum_{i=1}^m \phi_i\gamma_{i,m}$ *is the best approximation of* f *in* $B_{(m)}$, *then for each* $k = 1, \ldots, m$, *we have*

$$\lambda_k = \lim_{m\to\infty} \gamma_{k,m}.$$

Proof. Let $f \in B^2_\gamma(\Omega, Cl_n)$. As $\{\phi_k\}_k$ is Cl_n-complete in $B^2_\gamma(\Omega, Cl_n)$, we have that for each $\varepsilon > 0$, and $x \in \Omega$, $\|f(x) - \sum_{k=1}^m \phi_k(x)\gamma_{k,m}\|_{L^2_\Gamma} < \varepsilon$, for every $m > n_0 \in N$. In particular, from the constructions of the functions $\{\phi_i\}_i$, and approximating at each point y_j, for $j = 1, \ldots, m$, we have

$$\left\| f(y_j) - \sum_{k=1}^m \phi_k(y_j)\lambda_k \right\|_{L^2_\Gamma} < \varepsilon$$

for $m > n_0$. Which implies

$$\left\| \sum_{k=1}^m \phi_k(y_j)\gamma_{k,m} - \sum_{k=1}^m \phi_k(y_j)\lambda_k \right\|_{L^2_\Gamma} = \left\| \sum_{k=1}^m \phi_k(y_j)(\gamma_{k,m} - \lambda_k) \right\|_{L^2_\Gamma} < \varepsilon$$

for all $m > n_0 \in N$, and hence $\|\gamma_{k,m} - \lambda_k\|_{L^2_\Gamma} < \varepsilon$ for all $m > n_0 \in N$. That is $\gamma_{k,m} \to \lambda_k$ as $m \to \infty$. This proves the proposition. □

4. Application

Next, we see the application of the complete and minimal function systems that we constructed in approximating null solutions of first order partial differential equations of the in-homogeneous Dirac operator.

Proposition 4.1. *Let Ω be a bounded C^2-domain in $\mathbb{R}^n$ and $\Sigma = \partial\Omega$. Let also $g \in W^{2,k-1/2}_{\Gamma}(\Sigma, Cl_n)$, $k = 1, \dots$. Then the boundary value problem*

$$D_\gamma f = 0, \quad \text{in } \Omega \tag{4.1}$$

$$\mathrm{tr}_\Sigma f = g \tag{4.2}$$

has a unique solution $f \in W^{2,k}_{\Gamma}(\Omega, Cl_n)$ given by

$$f(x) = \int_\Sigma \Psi_\Gamma(x-y)\nu(y)g(y)d\Sigma_y, \quad x \in \Omega \tag{4.3}$$

Proof. As $g \in W^{2,k-1/2}_{\Gamma}(\Sigma, Cl_n)$, and tr_Σ is continuous, there exists a function $f \in W^{2,k}_{\Gamma}(\Omega, Cl_n)$ such that $g = \mathrm{tr}_\Sigma f$. Then from equation (2.8), we get the result. □

Proposition 4.2. *Let Ω, Σ and g be as in Proposition* 4.1*. Then for a given $\varepsilon > 0$ and for a given left γ-regular solution f given in equation* (4.3) *of the boundary value problem* (4.1), (4.2)*, there exist Clifford numbers β_j $(j = 1, \dots n_0)$ such that*

$$\left\| f - \sum_{j=1}^{n_0} \Psi^\Gamma_j \beta_j \right\|_{W^{2,k}_{\Gamma, Cl_n}} < \varepsilon \quad \text{on } \Omega.$$

Proof. Since the system $\Im$ is Cl_n-complete in the space of left γ-regular functions which are in $W^{2,k}_{\Gamma}(\Omega, Cl_n)$, the solution f of the boundary value problem (4.1), (4.2) can be approximated with finitely many elements of $\Im$. That means, $\exists \beta_j \in Cl_n$ $(j = 1, \dots, n_0)$ such that the above approximation inequality holds. The Clifford numbers β_j $(j = 1, \dots n_0)$ are determined by solving a system of equations obtained from the boundary conditions

$$\mathrm{tr}_\Sigma \sum_{j=1}^{n_0} \Psi^\Gamma_j \beta_j(y_i) = g(y_i)$$

for each $i = 1, \dots, n_0$, where $\{y_i : i = 1, \dots, n_0\}$ is a set of unisolvent points selected on Σ as in Proposition 3.3. □

Then a best approximation of the above solution can be obtained from the minimal functions.

Corollary 4.3. *Using the Cl_n-minimal functions $\{\phi_k\}_k$, the solution* (4.3) *of the BVP* (4.1), (4.2) *is approximated in the best way in*

$$B_{(n_0)} = \operatorname*{span}_{Cl_n}\left(\{\phi_j\}_{j=1}^{n_0}\right) \quad \text{as} \quad \left\| f - \sum_{j=1}^{n_0} \phi_j \lambda_j \right\|_{W^{2,k}_{\Gamma, Cl_n}} < \varepsilon$$

with λ_j $(j = 1, \ldots, n_0)$ determined as in Proposition 3.5.

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