The Intrinsic π-operator on Domain Manifolds in $\mathbb{C}^{n+1}$

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Abstract. The main theme of this paper is to study the hypercomplex π-operator over $\mathbb{C}^{n+1}$ via real, compact, $n+1$-dimensional manifolds called domain manifolds. We introduce an intrinsic Dirac operator for such types of domain manifolds and define an intrinsic π-operator, study its mapping properties and introduce a Clifford Beltrami equation in this context.

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1 Introduction

The π- operator is one of the tools used to study smoothness of functions and to solve some first order non-linear partial differential equations such as the Beltrami equation over domains in $\mathbb{C}$. The first attempt to generalize the π-operator using Clifford algebras was made by Shevchenko in 1962. It was W. Spröessig in 1979 [9] who introduced the π-operator over domains in $\mathbb{R}^{n+1}$ and developed some of its properties. Later it was re-introduced in a more general way by M.V. Shapiro and N.L. Vasilevski in 1992 [8]. See also [1, 2, 4] and elsewhere.

In [9] the Clifford π-operator is defined as the Clifford conjugate of the Dirac derivative of the Clifford valued Cauchy-Teodorescu transform. We shall adapt this definition to define a hypercomplex π-operator over domain manifolds in $\mathbb{C}^{n+1}$. While the term "domain manifold" arises in different contexts in mathematics such as in the context of variational problems in PDE and in works on Kähler manifolds the domain manifolds used here the same as those described in [5, 6, 7] and elsewhere. These are real, compact,

$n+1$-dimensional manifolds lying in $\mathbb{C}^{n+1}$. Any bounded domain in $\mathbb{R}^{n+1}$ with its boundary is an example of a domain manifold. Further, basic homotopy arguments are presented in [6] to show that there is an infinite abundance of many other domain manifolds lying in $\mathbb{C}^{n+1}$.

Such manifolds can be used to generalize Clifford analysis in $\mathbb{R}^{n+1}$ to an analysis in $\mathbb{C}^{n+1}$. This is done by defining an intrinsic Dirac operator for a domain manifold. This is done in [7]. Each domain manifold has a cell of harmonicity associated to it, see [6]. Cells of harmonicity are special types of domains in $\mathbb{C}^{n+1}$. These domain manifolds allow one to extend Clifford analysis to these cells of harmonicity. See for instance [5, 6, 7].

It is known that in Euclidean spaces, the $\pi-$operator is an operator of Calderon-Zygmund type. In this paper we study the generalized hypercomplex π-operator over domain manifolds in $\mathbb{C}^{n+1}$ and get some new and analogous results such as mapping properties, representation of it in terms of an integral and find spaces where it is invertible. Finally, we briefly look at the Clifford Beltrami equation over domain manifolds and its solvability using the intrinsic π-operator.

2 Preliminaries

Let $e_1, e_2, \ldots, e_n$ be orthonormal basis for $\mathbb{R}^n$. Consider the real 2^n-dimensional Clifford algebra Cl_n generated by the anticommutation relationship $e_ie_j + e_je_i = -2\delta_{i,j}e_0$ where e_0 is the identity of the algebra. It has as a basis

$$e_0, e_1, e_2, \ldots, e_n, \ldots, e_{j_1} \ldots e_{j_r}, \ldots, e_1e_2 \ldots e_n$$

with $j_1 < j_2 < \ldots j_r \leq n$. Thus each element of the algebra is represented in the form : $a = \sum_{A\subset\{1,\ldots,n\}} a_Ae_A$, where the a_A's are real numbers. In this regard every element x of $\mathbb{R}^{n+1}$ can be identified with an element $\sum\limits_{j=0}^{n} x_je_j$of Cl_n and therefore we have an embedding $\mathbb{R}^{n+1} \hookrightarrow Cl_n$. We also define the Clifford conjugate of $a = \sum_{A\subset\{1,\ldots,n\}} a_Ae_A \in Cl_n$ as $\sum_{A\subset\{1,\ldots,n\}} a_A\overline{e}_A$, where $\overline{e_{j_1} \ldots e_{j_r}} = (-1)^r e_{j_r} \ldots e_{j_1}$. We denote the conjugate of a by $\overline{a}$.

Definition 1 *For an element $a = \sum_A a_Ae_A \in Cl_n$, we define its Clifford norm by $\|a\|_0^2 = a\overline{a}_0$ where $a\overline{a}_0$ is the real part of $a\overline{a}$.*

The norm defined above satisfies $\| ab \|_0 \leq c(n) \| a \|_0 \| b \|_0$, where $c(n)$ is a dimensional constant. Every element x of $\mathbb{R}^{n+1}\backslash\{0\}$ is invertible with inverse $x^{-1}=\frac{\overline{x}}{\|x\|^2}$. We can complexify the real Clifford algebra Cl_n to get the complex Clifford algebra $Cl_n(\mathbb{C})$. It has complex dimension 2^n. A typical element of this algebra is of the form : $Z = e_0z_0 + e_1z_1 + e_2z_2 + \ldots + e_nz_n + \ldots z_{j_1\ldots j_r}e_{j_1\ldots j_r}\ldots + e_{1\ldots n}z_{1\ldots n}$, where $z_j = x_j + iy_j, (j = 0, 1, \ldots, n), \ldots,$ $z_{j_1j_2\ldots j_r} = x_{j_1j_2\ldots j_r} + iy_{j_1j_2\ldots j_r}, \ldots, z_{1\ldots n} = x_{1\ldots n} + iy_{1\ldots n}$, where $x_0, \ldots y_{1\ldots n} \in \mathbb{R}$. Here again every element $Z = (z_0, z_1, z_2, \ldots, z_n)$ of $\mathbb{C}^{n+1}$ is identified with the element $\sum_{j=0}^{n} z_je_j$ of $Cl_n(\mathbb{C})$. Therefore once again we have an embedding: $\mathbb{C}^{n+1} \hookrightarrow Cl_n(\mathbb{C})$. For $Z = X + iY \in Cl_n(\mathbb{C})$, $X, Y \in Cl_n(\mathbb{R})$ with $\| X \|_0^2 := x_0^2 + x_1^2 + \ldots + x_{1\ldots n}^2$, $\| Y \|_0^2 := y_0^2 + y_1^2 + \ldots y_{1\ldots n}^2$, we define the norm of Z as: $\| Z \|_{Cl_n(\mathbb{C})} := \sqrt{2^{n+1}}(\| X \|_0 + \| Y \|_0)$, see [5].

Unlike the case in the real Clifford algebra where non-zero elements are invertible, not all non-zero elements are invertible here in $Cl_n(\mathbb{C})$. For instance the element $z = e_0 + \sqrt{-1}e_j$,$(j = 1, \ldots, n)$ of $Cl_n(\mathbb{C})$ is a zero divisor in the algebra and hence not invertible.

Definition 2 *[7] For $w \in \mathbb{C}^{n+1}$, define the null cone at w to be the set $\{z \in \mathbb{C}^{n+1} : (z-w)\overline{(z-w)} = 0\}$ and denote it by $N(w)$. In particular, $N(0) = \{z \in \mathbb{C}^{n+1} : z\overline{z} = 0\}$.*

Definition 3 *For a domain $\Omega \subseteq \mathbb{C}^{n+1}$, a holomorphic function $f : \Omega \to Cl_n(\mathbb{C})$ is called a complex left monogenic function if $\Sigma_{j=0}^{n} e_j \frac{\partial f}{\partial z_j} = 0$ on Ω. So f satisfies the equations $\frac{\partial f}{\partial \overline{z}_j} = 0$ for $j = 1, \ldots, n$ and $\Sigma_{j=0}^{n} e_j \frac{\partial f}{\partial z_j}$. Similarly such a function is called complex right monogenic if $\sum_{j=0}^{n} \frac{\partial f}{\partial z_j} e_j = 0$ on Ω. The differential operator $\sum_{j=0}^{n} e_j \frac{\partial}{\partial z_j}$ is called the complex Dirac operator and it is denoted by $D_{\mathbb{C}}$.*

An example of a function which is both left and right complex monogenic is $\Psi(z) = \frac{1}{\omega_n}\frac{\overline{z}}{(z\overline{z})^{\frac{n+1}{2}}}$ for $z \notin N(0)$ and n odd. Here ω_n is the surface area of the unit sphere in $\mathbb{R}^{n+1}$. In this paper we shall assume that n is odd.

Definition 4 *[7] A compact real, $n+1$-dimensional C^1-manifold M lying in $\mathbb{C}^{n+1}$ is called a domain manifold if for all $z \in M$:*

1. *$N(z) \cap M = \{z\}$*

2. *$N(z)\cap TM_z=\{z\}$, where TM_z is the tangent space to the manifold M at the point z.*

In [5, 7] and elsewhere, it is shown that many aspects of real Clifford analysis are carried through to $\mathbb{C}^{n+1}$ via domain manifolds.

Definition 5 *[7] Suppose $M\subseteq\mathbb{C}^{n+1}$ is a domain manifold. Then the component of $\mathbb{C}^{n+1}\backslash\cup_{z\in\partial M}N(z)$ containing $int(M)$ is called a cell of harmonicity of M and it is denoted by M^+.*

If M is a domain manifold then its cell of harmonicity M^+ is a domain in $\mathbb{C}^{n+1}$.

Definition 6 *[7] Suppose M is a domain manifold. Then M is called a simple domain manifold if for each $w\in M$, there exist C^1- functions $\varphi_{j,w}:(-\frac{1}{2},\frac{1}{2})\to M$, with $(j=0,\dots,n)$ such that*
1: $\varphi_{j,w}$ is one to one.
2: $\varphi_{j,w}(0)=w$
3: $\varphi_{j,w}(t)=w+\lambda_j(t)e_j$ where $\lambda_j(t)\in\mathbb{C}$.

Examples of simple domain manifolds include bounded domains in $\mathbb{R}^{n+1}$. It is a simple matter to construct other nontrivial examples of simple domain manifolds. One can for instance take the closure of a bounded domain in $\mathbb{R}^{n+1}$ and consider all line segments in that set parallel to the x_j direction. Now one takes the same small homotopic deformation of these line segments in the complex planes parallel to the complex plane containing e_j. If the homotopy deformation is sufficiently small one obtains a simple domain manifold. See [7] for details.

The existence of the kind of paths given in Definition 6 enables one to define an intrinsic Dirac operator over domain manifolds in the following way.

Definition 7 *[7] Let M be a simple domain manifold. Then for a C^1-function $f:M\to Cl_n(\mathbb{C})$, the intrinsic Dirac derivative of f on M is defined as*

$$D_Mf(w):=\lim_{t\to 0}\sum_{j=0}^{n}e_j\frac{f(\varphi_{j,w}(t))-f(\varphi_{j,w}(0))}{\lambda_j(t)}.$$

If f is the restriction to M of a holomorphic function f^+,then $D_Mf=D_{\mathbb{C}}f|^+_{int(M)}$.

Definition 8 *On a simple domain manifold, M, we define the Bergman space of p-integrable functions as a right complex Clifford algebra module $B^p(M, Cl_n(\mathbb{C})) := \{f : M^+ \to Cl_n(\mathbb{C}) : f \in L^p(M, Cl_n(\mathbb{C}))$ and $D_{\mathbb{C}}f = 0\}$ where $p \in (1, \infty)$.*

We next define three important integral transforms over such domains lying in $\mathbb{C}^{n+1}$.

Definition 9 *[5, 7] Let M be a simple domain manifold and let $f : M \to Cl_n(\mathbb{C})$ be a C^1 function. Then the usual integral transforms are defined on M as follows*

1. $\zeta_M f(w) := \int\limits_M \Psi(z-w) f(z) dz^{n+1}$, $w \in M$
2. $\xi_{\partial M} f(w) := \int\limits_{\partial M} \Psi(z-w) Dz f(z)$, $w \notin \partial M$
3. $\xi'_{\partial M} f(w) := 2P.V. \int\limits_{\partial M} \Psi(z-w) DZ f(z)$, $w \in \partial M$

Here dz^{n+1} is the differential form $dz_0 \wedge \ldots \wedge dz_n$, $Dz = \Sigma_{j=0}^{n} (-1)^j e_j \hat{dz_j}$ with $\hat{dz_j} = dz_0 \wedge \ldots \wedge dz_{j-1} \wedge dz_{j+1} \wedge \ldots \wedge dz_n$. Further $\Psi(z-w) = \frac{1}{\omega_n} \frac{\overline{z-w}}{((z-w)(\overline{z-w}))^{\frac{n+1}{2}}}$ is the Cauchy kernel of the intrinsic Dirac operator defined over the domain manifold. As in real Clifford analysis, the integral transform given in (1) is the Teodorescu transform or Cauchy transform.

Theorem 1 *[7] Suppose M is a domain manifold and $f : M \to Cl_n(\mathbb{C})$ is a C^1 function with compact support then for all $w \in int(M)$, we have: $D_M \zeta_M f(w) = f(w)$.*

Remark 1 *One can see that if a C^∞-function $\psi : M \to Cl_n(\mathbb{C})$ has compact support, then the Teodorescu transform and the intrinsic Dirac operator are inverses of each other from both the left and right. That is $D_M \zeta_M \psi = \zeta_M D_M \psi = \psi$. See [5, 7] for further details.*

In [5] it is shown that if f is the restriction to M of a holomorphic function then $\zeta_M f$ extends to a holomorphic function in a neighborhood of M. If we denote the holomorphic extension of f by F then we shall denote the holomorphic extension of $\zeta_M f$ by $\zeta_M^\star F$.

Proposition 1 *[7]For $1 < p < \infty$, the Lebesgue space $L^p_M(Cl_n(\mathbb{C}))$ has a direct decomposition*

$$L^p_M(Cl_n(\mathbb{C})) = B^p(M, Cl_n(\mathbb{C})) \oplus \overline{D}_M(W_0^{p,1}(M, Cl_n(\mathbb{C})))$$

where $\oplus$ is a direct sum, and $W_0^{p,1}(M, Cl_n(\mathbb{C}))$ is a Sobolev space of $Cl_n(\mathbb{C})$ valued functions with compact support in M .

As usual the above decomposition gives us projection operators

$$P_M : L^p(M, Cl_n(\mathbb{C})) \to B^p(M, Cl_n(\mathbb{C}))$$

and

$$Q_M : L^p(M, Cl_n(\mathbb{C})) \to \overline{D}_M(W_0^{p,1}(M, Cl_n(\mathbb{C})))$$

with $Q_M = I_M - P_M$ where I_M is the identity operator on M.

Proposition 2 *Let $p \in (1, \infty)$. Then for $f \in L^p(M, Cl_n(\mathbb{C}))$, there exists a function $g \in W_0^{p,1}(M, Cl_n(\mathbb{C}))$ such that $D_M f = \Delta_M g$.*

Proof From the direct decomposition result, if f is a p-integrable complexified Clifford algebra valued function over the domain manifold M, then, $f = P_M f + Q_M f$ where, P_M is the Bergman projection operator and Q_M is its direct complement. Thus, $Q_M f \in \overline{D}_M(W_0^{p,1}(M, Cl_n(\mathbb{C})))$. Therefore, there exists $g \in W_0^{p,1}(M, Cl_n(\mathbb{C}))$ such that $Q_M f = \overline{D}_M g$ and hence $f = P_M f + \overline{D}_M g$. But $P_M f$ is a function in the Bergman space $B^p(M, Cl_n(\mathbb{C}))$ and therefore it is annihilated by D_M. Applying the intrinsic Dirac operator on both sides of the last equality above we have the result. □

3 The Intrinsic π-operator in $\mathbb{C}^{n+1}$

Here the generalized π-operator introduced in [9] is adapted to define an intrinsic π-operator over simple domain manifolds.

Definition 10 *Let M be a simple domain manifold and suppose $f : M \to Cl_n(\mathbb{C})$ is a C^1 function. The intrinsic hypercomplex π-operator on M, denoted by π_M, is defined as $\pi_M := \overline{D}_M \zeta_M$, where*

$$\overline{D}_M \zeta_M f(w) := \sum_{j=0}^{n} \overline{e}_j \lim_{t \to 0} \frac{\zeta_M f(\psi_{j,w}(0)) - \zeta_M f(\psi_{j,w}(t))}{\lambda_j(t)}.$$

If the function f is a restriction of some holomorphic function f^+ on the domain manifold M, then the above intrinsic hypercomplex π-operator will simply be $\pi_M f = \sum_{j=0}^{n} \overline{e}_j \frac{\partial \zeta_M^\star f(z)}{\partial z_j}$.

Theorem 2 *Suppose M is a simple domain manifold and $f : M \to Cl_n(\mathbb{C})$ is a C^1 function. Then the singular integral*

$$P.V. \int_M \overline{D}_{\mathbb{C})} \Psi(z-w) f(z) dz^{n+1}$$

is well defined for each $w \in int(M)$.

Proof First suppose that C is a sub domain of M and $w \in C$. Then

$$P.V. \int_M \overline{D}_{\mathbb{C}} \Psi(z-w) f(z) dz^{n+1} = \int\limits_{M \setminus C} [\overline{D}_{\mathbb{C}} \Psi(z-w)] f(z) dz^{n+1}$$

$$+ P.V. \int\limits_C \overline{D}_{\mathbb{C}} \Psi(z-w) f(z) dz^{n+1}.$$

Now let us place $f(z)$ equal to $(f(z) - f(w)) + f(w)$, and consider the integral $P.V. \int_C \overline{D}_{\mathbb{C}} \Psi(z-w)(f(z) - f(w)) dz^{n+1}$. Now as f is a C^1 function $f(z) - f(w) = D' f_w(z-w) + \epsilon(w)(z-w)$ where $D' f_w$ is the derivative of f at w. Consequently there is a non-zero real number $C(w)$ such that $\|f(z) - f(w)\| \leq C(w) \|z-w\|$. So

$$\|\overline{D} \Psi(z-w)(f(z) - f(w)\| \leq c C(w) \frac{1}{\|z-w\|^n}$$

for some $c \in \mathbb{R}^+$. It follows that

$$\|P.V. \int_C \overline{D}_{\mathbb{C}} \Psi(z-w)(f(z) - f(w)) dz^{n+1}\| \leq c' C(w) \int_C \frac{1}{\|z-w\|^n} \|dz^{n+1}\|.$$

for some $c' \in \mathbb{R}^+$.

Converting to polar coordinates we see that

$$\int_C \frac{1}{\|z-w\|^n} \|dz^n\| \leq c'' \int_0^{R(w)} dr$$

for some $c'' \in \mathbb{R}^+$ and $R(w) = sup\{\|z-w\| : z \in \partial C\}$. So $\| \int_C \overline{D}\Psi(z-w)(f(z)-f(w))dz^{n+1}\|$ is finite.

Now let us consider the singular integral $P.V. \int_C \overline{D}\Psi(z-w)f(w)dz^{n+1}$.

We now choose C to be sufficiently small so that it is a homotopic deformed of a disc C' lying in TM_w and with center at w. From conditions 1 and 2 of the definition of a domain manifold we know that such a deformation is possible and that the homotopy avoids the null cone $N(w)$ except at w. Further from [6] we know that this disc can be homotopically deformed avoiding $N(w)\backslash\{w\}$ to the disc $D(w,R)$ centered at w and of radius R lying in $\mathbb{R}^{n+1}+w$.

$\overline{D}\Psi(z-w)$ is homogeneous and the numerator of $\overline{D}\Psi(z-w)$ is a second order harmonic polynomial. Consequently $\int_{D(w,R)} \overline{D}\Psi(z-w)f(w)dx^{n+1} = 0$. Elementary continuity and homogeneity arguments now tell us that if C is sufficiently small then $\|P.V. \int_C \overline{D}\Psi(z-w)f(w)dz^{n+1} - P.V. \int_{D(w,R)} \overline{D}\Psi(z-w)f(w)dx^{n+1}\| < \epsilon$ for some $\epsilon \in \mathbb{R}^+$. The result follows. $\Box$

Following from this one can now deduce:

Theorem 3 *Suppose M is a simple domain manifold and $f : M \to Cl_n(\mathbb{C})$ is a C^1 function. Then*

$$\pi_M f(w) = P.V. \int_M \overline{D}\Psi(z-w)f(z)dz^{n+1} + \frac{n-1}{n+1}f(w). \tag{1}$$

Proof As in the previous theorem let us suppose that C is a sub domain of the manifold M and that $w \in C$. Then

$$\pi_M f(w) = \int_{M\backslash C} \overline{D}_{\mathbb{C}}\Psi(z-w)f(z)dz^{n+1} + \pi_C f(w).$$

As M, and C, are simple domain manifolds then for T sufficiently small the set $C_{j,t} = \{z + \varphi_{j,z}(t)e_j : z \in C\}$ is a sub domain of M for $t \in (0,T)$ and $j = 0, \ldots, n$.

Now consider

$$\Sigma_{j=0}^n \overline{e}_j \lim_{t\to 0} \frac{1}{t}(\int_C \Psi(z-w)f(z)dz^{n+1} - \int_{C_{j,t}} \Psi(z-w)f(z)dz^{n+1}).$$

This limit evaluates to

$$-\int_C \Sigma_{j=0}^n \overline{e}_j \Psi(z-w)f_j'(z)dz^{n+1} + \int_{\partial C} \overline{Dz}\Psi(z-w)f(z)$$

where $f_j'(z) = \lim_{t\to 0} \frac{(f((z+\varphi_{j,z}(t))-f(z)}{t}$ and $\overline{Dz} = \Sigma_{j=0}^{n}(-1)^j \overline{e}_j \hat{dz}_j$. Now $\int_C \Psi(z-w)f_j'(z)dz^{n+1}$ is a weakly singular integral and we can choose C so that the continuous function f_j' is bounded on C. Consequently for each $\epsilon > 0$ we can chose C sufficiently small so that $\| \int_C \Psi(z-w)f_j'(z)dz^{n+1}\| < \epsilon$. So we are left with the integral $\int_{\partial C} \overline{Dz}\Psi(z-w)f(z)$.

Now let us consider the integral $\int_{\partial C} \overline{Dz}\Psi(z-w)f(z)$. Again we place $f(z) = (f(z)-f(w))+f(w)$. Again via continuity arguments and the degree of homogeneity of $\Psi(z-w)$ we can choose C sufficiently small so that given $\epsilon > 0$

$$\| \int_{\partial C} \overline{Dz}\Psi(z-w)(f(z)-f(w))\| < \epsilon.$$

In [6] it is shown that for $n \geq 2$ the tangent space TM_w of a domain manifold can be deformed homotopically to $\mathbb{R}^{n+1}+w$ within $\mathbb{C}^{n+1}\backslash N(w)$. Consequently we may choose C so that it is a homotopy deformation of a rectangle R lying in $\mathbb{R}^{n+1}+w$, centered at w with normal vectors $\pm e_0, \ldots, \pm e_n$ and radius r. Now $\lim_{r\to 0} \Sigma_{j=0}^{n} \overline{e}_j \int_{E_j} \Psi(z-w)f(w)\hat{dx}_j = \frac{n-1}{n+1}f(w)$, where E_j are the two faces of ∂R with normal vectors $\pm e_j$. Further for R sufficiently small the continuity and degree of homogeneity of Ψ tells us that for each $\epsilon > 0$ we can choose C sufficiently small so that

$$\| \int_{\partial C} \Psi(z-w)f(w)d\sigma(z) - \int_{\partial R} \Psi(z-w)f(w)d\sigma(z)\| < \epsilon$$

where σ are the Lebesgue measures of ∂C and ∂R respectively. The result follows. $\square$

Suppose now that the function f appearing in the previous two theorems is the restriction to M of a holomorphic function $F(z)$ defined in a neighborhood of M. We have previously observed that in this case there is a holomorphic function G defined in a neighborhood of M satisfying $G|_M = \zeta_M f$ We denote G by $\zeta_M^\star F$. Further from earlier observations it may be determined that on this occasion $\pi_M f$ is equal to $\overline{D}_{\mathbb{C}}\zeta^\star F$ which is a holomorphic function. From Equation 1 we now have:

Theorem 4 *Suppose that M is a simple domain manifold and $f : M \to Cl_n(\mathbb{C})$ is the restriction to M of a holomorphic function $F(z)$ defined in an open neighborhood of M. Then P.V. $\int_M \overline{D}\Psi(z-w)f(z)dz^{n+1}$ is the restriction to M of a holomorphic function defined in a neighborhood of M.*

4 Mapping properties of the π_M-operator

We begin with:

Theorem 5 *For M a simple domain manifold the operator π_M is an isometry on $L^2(M, Cl_n(\mathbb{C}))$, the space of square integrable $Cl_n(\mathbb{C})$ valued functions defined on M.*

Proof Suppose that f and g are C^2 functions with compact support on $int(M)$ with values in $Cl_n(\mathbb{C})$. Then define the inner product $< f, g >$ to be $\int_M \overline{f}(z)g(z)dz^{n+1}$. Therefore, from this inner product we find that since f and g are C^2 and have compact support in $int(M)$ then from [7, 3] we know that $\zeta_M f$ and $\zeta_M g$ can be replaced by $\zeta_M f - F_1$ and $\zeta_M g - F_2$ where F_1 and F_2 are complex left monogenic functions in a neighborhood of $intM$, $F_1 = \zeta_M f$ on ∂M and $F_2 = \zeta_M g$ on ∂M. Consequently

$$\langle \pi_M f, \pi_M g \rangle_M = \langle \zeta_M f, D_M \overline{D}_M \zeta_M g \rangle_M.$$

As $D_M \overline{D}_M = \overline{D}_M D_M$

$$< \zeta_M f, \overline{D}_M D_M g > = < \zeta_M f, \overline{D}_M g > .$$

Again as f and g are continuously differentiable and have compact support in $int(M)$

$$< \zeta_M f, \overline{D}_M f > = < D_M \zeta_M f, g > = < f, g > .$$

A density argument now gives the result. $\square$

Let $\overline{B}^2(M, Cl_n(\mathbb{C}))$ denote the conjugate of the Bergman space $B^2(M, Cl_n(\mathbb{C}))$. Then we have :

Proposition 3 *For M a simple domain manifold $\pi_M : \overline{B}^2(M, Cl_n(\mathbb{C})) \to B^2(M, Cl_n(\mathbb{C}))$.*

Outline of Proof The result follows from noting that $D_M \zeta_M f = f$ and $\overline{D}_M D_M = D_M \overline{D}_M$. $\square$

Similarly $\pi_M : B^2(M, Cl_n(\mathbb{C})) \to \overline{B}^2(M, Cl_n(\mathbb{C}))$.

A similar and simple calculation also reveals that $\overline{\pi}_M \pi_M f = f$ and $\pi_M \overline{\pi}_M f = f$ whenever $f \in D_M(W_0^{2,1}(M, Cl_n(\mathbb{C}))$.

5 The Clifford Beltrami-Equation

We conclude by briefly discussing an analogue of the Beltrami equation in the context we have described here. The classical Beltrami equation $\frac{\partial f}{\partial z} - q\frac{\partial f}{\partial \bar{z}} = 0$, where $f, q : \Omega \subset \mathbb{C} \to \mathbb{C}$ are some measurable functions with $\parallel q \parallel < 1$, has been studied by many authors. This equation can also be studied in the higher dimensional complex space $\mathbb{C}^{n+1}$ via domain manifolds by considering the intrinsic Dirac operator D_M. In this setting we have a Clifford Beltrami-equation.

Definition 11 *Let M be a simple domain manifold and let $q : M \to Cl_n(\mathbb{C})$ be a measurable function. For $f \in W^{2,1}(M, Cl_n(\mathbb{C}))$, the Clifford Beltrami-equation over M is defined as*

$$D_M f = q\overline{D}_M f \tag{2}$$

Equation (2) gives another integral equation given by $f = \zeta_M h + \phi$ where $\phi \in \ker D_M(M, Cl_n(\mathbb{C}))$ and $h = q\overline{D}_M\phi$. Then applying the Clifford conjugate Dirac operator on both sides of this last integral equation, we have $\overline{D}_M f = \overline{D}_M\zeta_M h + \overline{D}_M\phi = \pi_M h + \widetilde{\phi}$, with $\widetilde{\phi} = \overline{D}_M\phi$. Therefore $D_M f = q\pi_M h + q\widetilde{\phi}$ with

$$h = q\pi_M h + q\widetilde{\phi} \tag{3}$$

The solvability of equation (2) is equivalent to the solvability of equation (3). But equation (3) can be studied by considering the map : $h \mapsto q\pi_M h$ on the Lebesgue space $L^2(M, Cl_n(\mathbb{C}))$. Under the assumption that $\parallel q \parallel < 1$, we see that the above map is a contraction and therefore it has a fixed point. That fixed point is the solution of 3. Therefore from this solution h we get a solution to the Clifford Beltrami equation. The solution is $f = \zeta_M q\pi_M h + \zeta_M q\widetilde{\phi} + \phi$ with $\phi \in \ker D_M(M, Cl_n(\mathbb{C}))$. Therefore we can state the following proposition.

Proposition 4 *The Clifford Beltrami equation (2) and equation (3) are equivalent.*

References

[1] K. Gürlebeck, *On some classes of* Π*-operators*, Dirac Operators in Analysis, J. Ryan and D. Struppa, (eds), Pitman, Harlow, 41-57, 1998.

[2] K. Gürlebeck and U. Kähler, *On a Spatial Generalization of the Complex π -Operator*, Journal of Analysis and its Applications, Vol.15, No.2, 1996, 283-297.

[3] K. Gürlebeck and W. Spröessig, *Quaternionic Analysis and Elliptic Boundary Value Problems*, Birkhäuser, Basel, 1990.

[4] D. A. Lakew, *Elliptic BVPs, Cl_n-Complete Function Systems and the π-Operator,* Ph.D. Thesis, University of Arkansas, Fayetteville, 2000.

[5] J. Ryan, *Applications of Complex Clifford Analysis to the Study of Solutions to Generalized Dirac and Klein-Gordon Equations with Holomorphic Potentials*, Journal of Differential Equations, 67, 1987, 295-329.

[6] J. Ryan, *Cells of harmonicity and generalized Cauchy integral formulas*, Proceedings of the London Mathematical Society, 60, 1990, 385-401.

[7] J. Ryan, *Intrinsic Dirac Operators in $\mathbb{C}^n$*, Advances in Mathematics 118, 99-133, 1996.

[8] M. V. Shapiro and N. L. Vasilevski, *On the Bergman Kernel Function in Hypercomplex Analysis*, Acta Applicandae Mathematicae, 46, No.1, 1997, 1-27.

[9] W. Spröessig, *Über eine Mehrdimensionale OperatarenrechnungÜber Beschränkten Gebienten des $\mathbb{R}^n$*, Thesis, TH Karl-Marx-Stadt, 1979.

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