# Some Exponentials of $D := \frac{d}{dx}$

**By**

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This work was done when I was at Virginia State University in 2008. I decided to send it for publication, because I thought it may motivate students, that looking the usual things differently, can generate totally different phenomenon. Indeed this note shows exactly that on the usual ordinary differential operator $D$ defined by $D := \frac{d}{dx}$. What happens when we exponentiate $D$? . What different and interesting properties can this action generate on smooth functions?. What happens to those rules of differentiations of calculus? . To answer few of these curiosities, we start from the very definition of exponentiating $D$ and extrapolate that to other varieties. I state results as examples and put some problems as exercises.

**Definition 1** $e^D := \sum_{n=0}^{\infty} \frac{D^n}{n!}$ , $e^{-D} := \sum_{n=0}^{\infty} \frac{(-1)^n D^n}{n!}$ .

Therefore for a function $f \in C^{\infty}(I \subseteq \mathbb{R})$ , where $I$ is some open interval , we define the exponential derivative of $f$ at a point $x \in I$ as follows:

**Definition 2** $e^D(f(x)) := \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}$, *where* $f^{(n)}(x) = D^n(f(x))$.

Let us see how the above definitions work for some infinitely differentiable functions.

**Example 3** *The function* $f(x) = e^x$ *is a* $C^{\infty}$*-function over* $\mathbb{R}$ *and*

$e^D(e^x) = e^{x+1}$

**Proof.** From the definition

$$e^D(f(x)) = e^D(e^x) = \sum_{n=0}^{\infty} \frac{D^n(e^x)}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{e^x}{n!} = e^x \sum_{n=0}^{\infty} \frac{1}{n!}$$

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But the infinite sum $\sum_{n=0}^{\infty} \frac{1}{n!}$ is $e$.

Therefore,

$$e^D(e^x) = e^x.e = e^{x+1}.$$

■

Some more examples are given below.

**Example 4** *1.* $e^D\left(e^{-x}\right) = e^{-x-1}$

*2.* $e^{-D}\left(e^x\right) = e^{x-1}$

**Example 5** $e^D\left(\sin x\right) = \sin\left(x+1\right)$

***Proof.*** *The function* $f(x) = \sin x \in C^{\infty}(\mathbb{R})$. *Therefore,*

$$\begin{aligned} e^D(\sin x) &= \sum_{n=0}^{\infty} \frac{D^n(\sin x)}{n!} \\ &= \sin x \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}}_{\substack{\| \\ \cos 1}} + \cos x \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}}_{\substack{\| \\ \sin 1}} \end{aligned}$$

$$= \sin x \cos 1 + \cos x \sin 1$$

*But*

$$\sin x \cos 1 + \cos x \sin 1 = \sin(x+1)$$

*and that proves the example.* ■

Similar procedures will provide the following examples:

**Example 6**

**Example 7** *1.* $e^D\left(\cos x\right) = \cos\left(x+1\right)$

*2.* $e^D\left(x^2\right) = x^2 + 2x + 1$

**Properties of** $e^D$ :

1. $e^D\left(k\right) = k$
2. $e^D\left(kf\right) = ke^D\left(f\right)$

3. $e^D(f+g) = e^D(f) + e^D(g)$

**Definition 8** *Define the sine hyperbolic and cosine hyperbolic of D as :*

$$\sinh D := \frac{e^D - e^{-D}}{2}$$

*and*

$$\cosh D := \frac{e^D + e^{-D}}{2}.$$

**Example 9**

1. $\sinh D(e^x) = \frac{(e-1)}{2}e^{x-1}$
2. $\sinh D(\sinh x) = \frac{e-1}{2e}\cosh x$
3. $\sinh D(\cosh x) = \frac{e^2-1}{2e}\sinh x$

**Proof.** Left as exercises. ■

**Problem 10** *Find the following hyperbolic derivatives:*

1. $\cosh D(\sinh x)$
2. $\cosh D(\cosh x)$
3. $\cosh D(e^{-x})$

**Problem 11** *For the $n^{th}$ degree polynomial : $p_n(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ , find $e^D(p_n(x))$.*

**Other exponentials of** $D$ :

**Definition 12** *For $a(>0, \neq 1) \in \mathbb{R}$ , define $a^D := e^{D\ln a}$*

**Example 13** $a^D(e^x) = ae^x$

***Proof.*** *Using the above defined new derivative,*

$$a^D(e^x) = e^{D\ln a}(e^x) = \sum_{n=0}^{\infty} \frac{(D\ln a)^n e^x}{n!}$$

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{(\ln a)^n D^n e^x}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(\ln a)^n e^x}{n!} = e^x \sum_{n=0}^{\infty} \frac{(\ln a)^n}{n!} \end{aligned}$$

*But the expression* $\sum_{n=0}^{\infty} \frac{(\ln a)^n}{n!}$ *is a and this proves the claim* ■

One can easily show that $\left(a^D\right)^n(e^x) = a^n e^x$
Similarly, one can prove the following results.

**Example 14** *1.* $2^D(e^x) = 2e^x$

*2.* $2^{-D}(e^x) = \left(\frac{1}{2}\right)^D(e^x)$

Finally let us prove one more result:

**Claim 15** $a^D(\sin x) = \sin(x + \ln a)$

**Proof.** Using again the definition above, we have

$$\begin{aligned} a^D(\sin x) &= \sum_{n=0}^{\infty} \frac{(\ln a)^n D^n(\sin x)}{n!} \\ &= \sin x \left( \sum_{n=0}^{\infty} \frac{(-1)^n (\ln a)^{2n}}{(2n)!} \right) + \cos x \left( \sum_{n=0}^{\infty} \frac{(-1)^n (\ln a)^{2n+1}}{(2n+1)!} \right) \end{aligned}$$

$$\begin{aligned} &= \sin x \cos(\ln a) + \cos x \sin(\ln a) \\ &= \sin(x + \ln a) \end{aligned}$$

■

**Corollary 16** $e^D(\sin x) = \sin(x+1)$

**Corollary 17** *For* $n \in \mathbb{N}$, $e^{nD}(\sin x) = \sin(x+n)$, *where* $e^{nD} = \left(e^D\right)^n$, *the n-th power of the exponential derivative* $e^D$.

**Problems to look at**: What is the action of $e^D$ on products and quotients of functions? This is a good exercise to look at.