Computational Methods and Function Theory
Volume 2 (2002), No. 1, 215–228

Complete Function Systems and Decomposition Results Arising in Clifford Analysis

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(Communicated by Helmuth Malonek)

Abstract. For Ω a sufficiently smooth unbounded domain in $\mathbb{R}^n$ we develop a decomposition result for the Sobolev space $W^{pl-1}_{\mathcal{C}\ell_{0,n}}(\Omega)$. We also use modified Cauchy-Green type kernels to construct Clifford analytic-complete function systems in the generalized Bergman space $B^{p,l}_{\mathcal{C}\ell_{0,n}}(\Omega) := \ker D^l(\Omega) \cap W^{p,l-1}_{\mathcal{C}\ell_{0,n}}(\Omega)$, where D^l is the l-th iterate of the Dirac operator, l is a positive integer less than n and $n/(n-l+1) < p < \infty$. The modified Cauchy-Green kernels ensure that p lies in this range. Without the modification of the kernels one is restricted to a smaller range. These functions are used to approximate solutions of the equation $\triangle^k u = 0$ with some boundary conditions and with $2k < n$. Some similar results are presented for sufficiently smooth unbounded domains lying in hyperbolas.

Keywords. Clifford Analysis, complete function systems, Bergman spaces, decomposition spaces, elliptic boundary value problems, Dirac operators and hyperbolas.

2000 MSC. 30G35, 35C15, 35F15, 35J05, 30E20, 35A35.

1. Introduction

This paper is a continuation of [10] and builds on results developed in [3, 4, 5, 6, 7, 9, 11, 12, 13, 16, 17, 18]. See also [1, 21].

Boundary value problems of linear and non-linear partial differential equations have long been solved by analytic and approximation techniques. Clifford analysis techniques in solving boundary value problems have been used increasingly by a number of authors. See for instance the papers cited in the previous paragraph. In this paper Clifford analytic complete systems of functions are constructed in the function space $B^{p,l}_{\mathcal{C}\ell_{0,n}}(\Omega)$, where $B^{p,l}_{\mathcal{C}\ell_{0,n}}(\Omega)$ is the generalized Bergman p-space defined in terms of the l-th power of a Dirac operator D and Ω is an unbounded

Received January 24, 2002, in revised form November 22, 2002.

ISSN 1617-9447/$ 2.50

domain in $\mathbb{R}^n$. To achieve this we also present a direct decomposition of the Sobolev space $W^{p,l-1}_{\mathcal{C}\ell_{0,n}}$ space as

$$W^{p,l-1}_{\mathcal{C}\ell_{0,n}}(\Omega) = B^{p,l}_{\mathcal{C}\ell_{0,n}}(\Omega) \oplus D^l(W^{0,p,2l-1}_{\mathcal{C}\ell_{0,n}}(\Omega)),$$

where $\oplus$ is a direct sum $l < n$ and $n/(n-l+1) < p < \infty$. When $p = 2$ and $l = 1$ this corresponds to the L^2 decomposition for bounded Liapunov domains described in [6]. We need the condition that $l < n$ as we use modified Cauchy-Green kernels to obtain our results. These modified kernels possess a cancellation property provided $l < n$. This cancellation property is needed in order that our results hold for the range $n/(n-l+1) < p < \infty$. Without the cancellation property we would be forced to work on a more limited range for p.

We apply our results to solve certain types of boundary value problems including for the differential equation $\triangle^l u = 0$ for $l < n$. We introduce some complete functions spaces which enable us to approximate solutions to these boundary value problems provided $2l < n$.

In [17, 18], Ryan has also developed function theories for the spherical and hyperbolic Dirac operators which enable one to set up different types of boundary value problems on spheres and hyperbolas. This is done by transferring theories of Clifford analysis developed over the Euclidean space $\mathbb{R}^n$ to spheres and hyperbolas through Möbius transformations. These diffeomorphisms preserve monogenicity. In particular these transformations preserve Cauchy's Theorem and the Cauchy integral formula. See also Sudbery [19], Bojarski [2] and Peetre and Qian [14].

As we lack Cauchy-Green type kernels for certain types of Dirac operators in the setting of the hyperbola we shall restrict attention here to the case $l = 1$. In this case though we are able to establish a decomposition of the L^p space for an unbounded domain in terms of a Bergman space and a Sobolev space for suitable unbounded domains on the hyperbola. As the Laplacian for the sphere and hyperbola has two different factorizations in terms of Dirac operators we are able to establish two different types of decompositions for two different types of Bergman spaces. For one of these Dirac operators we are able to establish a complete function system for suitable unbounded domains.

2. Preliminaries

A real 2^n-dimensional algebra in which $\mathbb{R}^n$ is embedded so that the multiplication defined there satisfies the relationship: $x^2 = -\|x\|^2$ for each $x \in \mathbb{R}^n$, is called a Clifford algebra. We denote it by $\mathcal{C}\ell_{0,n}(R)$. If $e_1, e_2, \ldots, e_n$ form an orthonormal basis of $\mathbb{R}^n$, then from the above defining multiplication, we have

$$e_i e_j + e_j e_i = -2\delta_{ij}$$

where δ_{ij} is the Kroneker delta. A general basis element for this algebra is $e_{j_1}\cdots e_{j_r}$ where $j_1 < \cdots < j_r$ and $1 \le r \le n$. When $n = 1$ the Clifford algebra is the complex field. The analogue of complex conjugation is the operator

$$-: \mathcal{C}\ell_{0,n} \to \mathcal{C}\ell_{0,n}, \qquad -(e_A) = (-1)^r e_{j_r}\cdots e_{j_1}.$$

We usually write $\overline{a}$ for $-(a)$ where a is an arbitrary point in $\mathcal{C}\ell_{0,n}$.

Every non-zero vector x of $\mathbb{R}^n$ is invertible and its inverse is given by

$$x^{-1} = \frac{-x}{\|x\|^2}.$$

This inverse is called the Kelvin inverse of x.

For every element $a \in \mathcal{C}\ell_{0,n}(R)$, we define the Clifford norm of a to be

$$\|a\| = \left(\sum {a_A}^2\right)^{1/2}.$$

In fact the real part, or identity component, of $a\overline{a}$ is $\|a\|^2$.

Besides conjugation we shall also need the operator

$$\sim: \mathcal{C}\ell_{0,n} \to \mathcal{C}\ell_{0,n}, \qquad \sim(e_{j_1}\cdots e_{j_r}) = e_{j_r}\cdots e_{j_1}.$$

Again we will write $\tilde{a}$ for $\sim(a)$.

Besides the algebra $\mathcal{C}\ell_{0,n}$, we shall also use the algebra $\mathcal{C}\ell_{1,n}$ which is generated from the Minkowski-Krain space $R^{1,n}$. This algebra is generated by the orthonormal basis $e_1, \ldots, e_n, f_{n+1}$ of $R^{n,1}$, where f_{n+1} satisfies the relationships

$${f_{n+1}}^2 = 1, \qquad f_{n+1}e_j = -e_j f_{n+1} \quad \text{for } 1 \le j \le n.$$

In this case $\mathcal{C}\ell_{1,n} = \mathcal{C}\ell_{0,n} \oplus \mathcal{C}\ell_{0,n}f_{n+1}$. For a and $b \in \mathcal{C}\ell_{0,n}$ we define the norm of $c = a + f_{n+1}b \in \mathcal{C}\ell_{1,n}$ to be $(\|a\|^2 + \|b\|^2)^{1/2}$ and we denote it again by $\|c\|$. The operators $-$ and $\sim$ readily extend to the algebra $\mathcal{C}\ell_{1,n}$. See [15] for details. More details on Clifford algebras can be found in [15] and elsewhere.

Some main function spaces considered in this paper are $\mathcal{C}\ell_{0,n}$ Sobolev-Slobodeckij spaces.

The following three definitions are standard, see for instance [20]. We include them for completeness.

Definition 1. *Suppose Ω is a domain in $\mathbb{R}^n$. Let $\phi \in C^{\infty}_{\mathcal{C}\ell_{0,n}}(\Omega)$, the space of $\mathcal{C}\ell_{0,n}$ valued C^∞ functions defined on Ω. Then*

$$C^{0,\infty}_{\mathcal{C}\ell_{0,n}} := \left\{ f \in C^\infty : \operatorname{supp} f \subset K^{\text{compact}} \subset \Omega \right\},$$

where $\operatorname{supp}\phi$*, the support of ϕ, is the closure of the set $\{x \in \Omega : \phi(x) \ne 0\}$.*

Definition 2. *A locally integrable $\mathcal{C}\ell_{0,n}$ valued function f defined on Ω has a locally integrable, weak or distributional derivative of order r, denoted by $\partial^r f$ if there is a locally integrable $\mathcal{C}\ell_{0,n}$ valued function g such that*

$$\int_\Omega f(x)\partial^{\|r\|}\phi(x)dx^n = (-1)^{\|r\|}\int_\Omega g(x)\phi(x)dx^n$$

for all $\phi \in C^{0,\infty}_{\mathcal{C}\ell_{0,n}}(\Omega)$, *where*

$$\partial^{\|r\|} = \frac{\partial^{r_1}}{\partial x_1^{r_1}} \cdots \frac{\partial^{r_n}}{\partial x_n^{r_n}}$$

with $r_1 + \cdots + r_n = \|r\| \in \mathbb{N}$.

Definition 3. *The Sobolev space* $W^{p,m}_{\mathcal{C}\ell_{0,n}}(\Omega)$, *for* $1 < p < \infty$, *is defined to be the Banach space*

$$\left\{ f\colon \Omega \to \mathcal{C}\ell_{0,n} :\ \partial^{\|r\|} f \ \textit{exists and is } L^p \textit{ integrable for } 0 \le \|r\| \le m \right\}$$

with norm

$$\|f\|_{W^{p,m}(\Omega)} := \left(\sum_{0 \le \|r\| \le m} \|\partial^{\|r\|} f\|_p{}^p \right)^{1/p}.$$

The space $W^{0,p,m}_{\mathcal{C}\ell_{0,n}}(\Omega)$ is the completion of $C^{0,\infty}_{\mathcal{C}\ell_{0,n}}(\Omega)$ in the space $W^{p,m}_{\mathcal{C}\ell_{0,n}}(\Omega)$. This is the same as the space

$$\left\{ f \in W^{p,m}_{\mathcal{C}\ell_{0,n}}(\Omega) :\ \mathrm{tr}_{\partial\Omega}\, D^k f = 0,\ 0 \le k < m \right\}.$$

Remark. The space $W^{p,0}_{\mathcal{C}\ell_{0,n}}(\Omega)$ coincides with $L^p_{\mathcal{C}\ell_{0,n}}(\Omega)$, the space of p integrable $\mathcal{C}\ell_{0,n}$ valued functions on Ω. When $s > 0$ is not an integer, and $1 < p < \infty$, then the function space $W^{p,s}_{\mathcal{C}\ell_{0,n}}(\Omega)$ is called the Slobodeckij space. For detailed information on such spaces, see [20]. The Slobedeckij spaces are closely related to the investigation of boundary values of functions which belong to some Sobolev spaces which, in our case, are spaces of traces of functions from some Sobolev spaces. The trace operator is the one which is continuous and has the mapping property

$$\mathrm{tr}_{\partial\Omega}\colon W^{p,m}_{\mathcal{C}\ell_{0,n}}(\Omega) \to W^{p,m-1/p}_{\mathcal{C}\ell_{0,n}}(\partial\Omega)$$

and coincides with taking a non-tangential boundary limit whenever both are meaningful. See [20] for more details.

Analogues of these definitions and constructions may readily be developed for domains lying in hyperbolas.

3. Higher order Dirac operators and some of their applications

In this section, we study the function theory of the higher order iterate D^l of the Dirac operator

$$D = D_x = \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j},$$

where l is a positive integer less than n. To start with let $\Psi_1^z(x,y)$ be the Cauchy kernel for the Dirac operator over unbounded domains. Then

$$\Psi_1^z(x,y) = D_x \frac{1}{(2-n)\omega_n}\left(\frac{1}{\|x-y\|^{n-2}} - \frac{1}{\|x-z\|^{n-2}}\right).$$

Doing this procedure repeatedly, we can get functions $\Psi_l^z(x,y)$ on $\mathbb{R}^n \backslash \{y,z\}$ such that

$$\Psi_{l-1}^z(x,y) = D_x \Psi_l^z(x,y).$$

These functions are given iteratively by

$$\Psi_l^z(x,y) = \begin{cases} c_{n,l}\left(\dfrac{x-y}{\|x-y\|^{n-l+1}} - \dfrac{x-z}{\|x-z\|^{n-l+1}}\right) & \text{for } l \text{ odd,} \\ c_{n,l}\left(\dfrac{1}{\|x-y\|^{n-l+1}} - \dfrac{1}{\|x-z\|^{n-l+1}}\right) & \text{for } l \text{ even and } l < n. \end{cases}$$

The constants $c_{n,l}$ are chosen so that the recurrence relation

$$D_x \Psi_l^z(x,y) = \Psi_{l-1}^z(x,y)$$

holds and $D^l \Psi_l^z(x,y) = 0$, for $x \neq y$.

Definition 4. *A function $f\colon \Omega \subset \mathbb{R}^n \to \mathcal{C}\ell_{0,n}$ is called left l monogenic if $D^l f = 0$ and a function $g\colon \Omega \subset \mathbb{R}^n \to \mathcal{C}\ell_{0,n}$ is called right l monogenic if $gD^l = 0$.*

Example. The function $\Psi_1^z(x,y)$ is both a left and right monogenic function.

Basic properties of l-monogenic functions including Cauchy-Green type integral formulas and Borel-Pompeiu formulas are developed in [16].

Lemma 1. *Assume that $\|x\| > 2\max(\|y\|, \|z\|)$. Then*

$$\left|\frac{x-y}{\|x-y\|^{n-l+1}} - \frac{x-z}{\|x-z\|^{n-l+1}}\right| \leq \frac{C_n\|y-x\|}{\|x\|^{n-l+1}},$$

$$\left|\frac{1}{\|x-y\|^{n-l}} - \frac{1}{\|x-z\|^{n-l}}\right| \leq \frac{C_n\|y-z\|}{\|x\|^{n-l+1}}$$

for some dimensional constant $C_n \in \mathbb{R}^+$.

The proof follows standard arguments. See for instance [3] and elsewhere.

From the above lemma, we have the following proposition.

Proposition 1. *For a positive integer l, where $l < n$, the function*

$$\Psi_l^z(x,y) \in L^p_{\mathcal{C}\ell_{0,n}}(\Omega_\varepsilon)$$

for $n/(n-l+1) < p < \infty$, and $\Omega_\varepsilon = \Omega \backslash B(y,\varepsilon)$, where $B(y,\varepsilon)$ is a ball centered at y with radius ε.

Remark. Working with the unmodified kernel $E_l(x-y)$ of the iterate D^l of the Dirac operator has the limitation that over unbounded domains in $\mathbb{R}^3$ this kernel is not L^2 bounded on Ω_ε. Indeed for $n = 3$ the kernel $E_l(x-y)$ only belongs to

$L^p_{C\ell_{0,n}}(\Omega_\varepsilon)$ for $p \in (3,\infty)$. Furthermore it may easily be determined that in all dimensions $E_l(x-y)$ belongs to $L^p_{C\ell_{0,n}}(\Omega_\varepsilon)$ for $l < n$ for a more limited choice of p whenever Ω is unbounded. But since the modified kernel $\Psi^z_l(x,y) \in L^p_{C\ell_{0,n}}(\Omega_\varepsilon)$, for $3/2 < p < \infty$, when $n = 3$, we can use the modified kernel to overcome this limitation.

Proposition 2. *Let $n/(n-l+1) < p < \infty$, $k \in N \cup \{0\}$, and $f \in W^{p,k}_{C\ell_{0,n}}(\Omega)$ be a left l-monogenic function over Ω. Then there exist functions $f_i \in \ker D(\Omega) \cap W^{p,k+1-i}_{C\ell_{0,n}}(\Omega)$, $i = 1, 2, \dots, l$ such that*

$$f = \sum_{i=1}^{l} T^{\star(i-1)}_{\Omega} f_i,$$

where $T^{\star i}_{\Omega} u$ is the i-th modified Cauchy-Teodorescu transform $\int_\Omega \Psi^z_i(x,y)u(y)dy^n$.

Proof. The proof follows from the fact that null solutions of the Dirac operator are monogenic functions and from the fact that the solution of $Du = f$ in Ω is given by the generalized Teodorescu integral operator $u = T^{\star 1}_{\Omega} f$. ■

Corollary 1. *If f is left monogenic over Ω, then the $T^{\star 1}_{\Omega}$ transform of f is 2-monogenic or harmonic over Ω.*

Definition 5. *The Bergman p,l-space is defined to be* $\ker D^l(\Omega) \cap W^{p,l-1}_{C\ell_{0,n}}(\Omega)$ *and it is denoted by $B^{p,l}_{C\ell_{0,n}}(\Omega)$.*

Theorem 1. *The space $B^{p,l}_{C\ell_{0,n}}(\Omega)$ is complete.*

Proof. Suppose $y \in \Omega$ and $f \in B^{p,l}_{C\ell_{0,n}}$. Then from [16] it follows that for each closed ball $\overline{B}(y,r) \subset \Omega$

$$f(y) = \frac{1}{\omega_n} \int_{\partial B(y,r)} \sum_{j=1}^{l-1} E_j(x-y) n(x) D^{l-j} f(x)\, d\sigma(x).$$

It follows that

$$\frac{1}{2} r f(y) = \frac{1}{\omega_n} \int_{A(y,r,r/2)} \sum_{j=1}^{l-1} E_j(x-y) \frac{(x-y)}{\|x-y\|} D^{l-j} f(x)\, dx^n,$$

where $A(y,r,r/2)$ is the annulus or spherical shell $\{x \in \mathbb{R}^n : r/2 < \|x\| < r\}$ and ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. The result follows by replacing f by a Cauchy sequence $\{f_m\}_{m=1}^{\infty}$ in $B^{p,l}_{C\ell_{0,n}}(\Omega)$ and applying Hölder's inequality to the right side of the previous expression. ■

The following result is a decomposition of $W^{p,l-1}_{C\ell 0,n}(\Omega)$ as a sum of two closed subspaces formed using the iterated Dirac operator D^l.

Proposition 3. *Let $n/(n-l+1) < p < \infty$, and Ω be an unbounded C^2 domain which is also Lipschitz continuous. Then for each $l = 1, 2, \ldots, n$, we have*

$$W^{p,l-1}_{\mathcal{C}\ell_{0,n}}(\Omega) = B^{p,l}_{\mathcal{C}\ell_{0,n}}(\Omega) \oplus D^l(W^{0,p,2l-1}_{\mathcal{C}\ell_{0,n}}(\Omega)).$$

The proof follows very similar lines to the proof for the case $l = 1$ given in [8]. See also [20].

Proof. Let $f \in B^{p,l}_{\mathcal{C}\ell_{0,n}}(\Omega) \cap D^l(W^{0,p,2l-1}_{\mathcal{C}\ell_{0,n}}(\Omega))$. Then $D^l f = 0$ and $f = D^l g$ for some function $g \in W^{0,p,2l-1}_{\mathcal{C}\ell_{0,n}}(\Omega)$. But then $D^l f = D^l D^l g = (-1)^l \bigtriangleup^l g = 0$. This implies that $g = 0$, as $\bigtriangleup$ is invertible on $W^{0,p,k}_{\mathcal{C}\ell_{0,n}}(\Omega)$ for $k \geq 0$. Therefore $f = 0$.

Now consider a general element $f \in W^{p,l-1}_{\mathcal{C}\ell_{0,n}}(\Omega)$. Then taking $h_0 = \bigtriangleup_0^{-l} D^l f \in W^{0,p,2l-1}_{\mathcal{C}\ell_{0,n}}(\Omega)$, we have that $D^l h_0 \in D^l(W^{0,p,2l-1}_{\mathcal{C}\ell_{0,n}}(\Omega))$ and setting $g = f - D^l h_0$ we see $D^l g = D^l(f - D^l h_0) = 0$. So, $f = g + D^l h_0$. The uniqueness of this decomposition now follows from the argument given in the first paragraph of this proof. Thus the result follows. ■

We will also need the following projection operator

$$Q_l \colon W^{p,l-1}_{\mathcal{C}\ell_{0,n}}(\Omega) \to D^l W^{0,p,2l-1}_{\mathcal{C}\ell_{0,n}}(\Omega).$$

We conclude this section with some applications to boundary value problems.

Proposition 4. *Let $1 < p < \infty$, $k \geq 2$ and $f \in L^p_{\mathcal{C}\ell_{0,n}}(\Omega)$. Then*

$$u = T^{\star 1}_{\Omega} Q_1 \cdots T^{\star 1}_{\Omega} f$$

is a solution to the equation $D^k u = f$ and $\mathrm{tr}_{\partial\Omega} D^j u = 0$ for $j = 0, \ldots, l-2$.

The previous result only makes use of the case $l = 1$. Consider now the following.

Proposition 5. *Let $n/(n-l+1) < p < \infty$ and $f \in W^{p,l-1}_{\mathcal{C}\ell_{0,n}}(\Omega)$. Then*

$$u = T^{\star l}_{\Omega} Q_l T^{\star l}_{\Omega} f$$

is a solution to the equation $\bigtriangleup^l u = f$ and $\mathrm{tr}_{\partial\Omega} u = 0$, provided $l < n$.

The case $l = 1$ is covered in [7, 8]. As any function $g \in W^{p,3l-1-1/p}_{\mathcal{C}\ell_{0,n}}(\partial\Omega)$ has an extension $h \in W^{p,3l-1}_{\mathcal{C}\ell_{0,n}}(\Omega)$ then the function $u_1 = h - T^{\star l}_{\Omega} Q_l T^{\star l}_{\Omega} \bigtriangleup^l h$ is a solution to the equation $\bigtriangleup^l u = 0$ and $tr_{\partial\Omega} u_1 = g$ provided $2l < n$. Again the special case $l = 1$ is covered in [7, 8].

4. $\mathcal{C}\ell_{0,n}$-complete systems in $B^{p,l}_{\mathcal{C}\ell_{0,n}}(\Omega)$

In this section, we investigate functions dense in the space $L^p_{\mathcal{C}\ell_{0,n}}(\Omega) \cap \ker D^l(\Omega)$, where $n/(n-l+1) < p < \infty$. In order to be reasonably self-contained let us recall (see [6]) some generalizations of concepts known in classical analysis.

Definition 6. *Let V be a normed right-vector space over $\mathcal{C}\ell_{0,n}$. A system of points $\{x_m : m \in \mathbb{N}\} \subset V$ is called $\mathcal{C}\ell_{0,n}$ complete system in V, if the points approximate V finitely. i.e. for each $\epsilon > 0$, for each $x \in V$, there exists $c_i \in \mathcal{C}\ell_{0,n}$, $i = 1, 2, \ldots, n_0$ such that*

$$\left\| x - \sum_{i=1}^{n_0} x_i c_i \right\|_X < \epsilon.$$

Definition 7. *A system of points $\{x_m : m \in \mathbb{N}\} \subset V$ is called closed in V if every bounded $\mathcal{C}\ell_{0,n}$-valued right-linear functional F that vanishes on the points vanishes on the whole space V.*

Lemma 2. *The system of points $\{x_m : m \in \mathbb{N}\} \subset V$ is closed if and only if it is $\mathcal{C}\ell_{0,n}$-complete in V.*

Proposition 6. *Suppose that $n/(n-l+1) < p < \infty$ and Ω is an unbounded domain in $\mathbb{R}^n$, and Ω has a C^2 boundary. Suppose also that $X = \{x_m : m \in N\}$ is a dense set in the complement of Ω, then the double-indexed system of functions given by*

$$\{\Psi_m^z(x_k, x)\}_{m=1,k=0}^{\infty,l-1}$$

is $\mathcal{C}\ell_{0,n}$ complete in the space $B^{p,l}_{\mathcal{C}\ell_{0,n}}(\Omega)$.

Proof. Suppose that F is a bounded linear functional from the dual of $W^{p,l-1}_{\mathcal{C}\ell_{0,n}}(\Omega)$, and is such that $F(\Psi_m^z(x_k, x)) = 0$ for each $x_k \in X$. Then by the Riesz Representation Theorem there is a $\mathcal{C}\ell_{0,n}$ valued measure μ with support $[\mu]$ lying in Ω and

$$F(g) = \int_{[\mu]} d\mu(x) g(x)$$

for each $g \in W^{p,l-1}_{\mathcal{C}\ell_{0,n}}(\Omega)$. Suppose now that $f \in B^{p,l}_{\mathcal{C}\ell_{0,n}}(\Omega')$ where Ω' is a domain which contains the closure of Ω. Moreover Ω' has an non-empty open subset in its complement and z lies in this open set. Furthermore $\partial\Omega'$ is C^2, and f is left monogenic in a neighbourhood of the closure of Ω'. Then by the Cauchy-Green integral formula for unbounded domains

$$F(f) = \int_{[\mu]} d\mu(x) \frac{1}{\omega_n} \int_{\partial\Omega'} \sum_{m=1}^{l-1} \Psi_m^z(y, x) n(y) f(y)\, d\sigma(y).$$

As X is a dense subset of the complement of Ω it follows that $X \cap \partial\Omega'$ is a dense subset of $\partial\Omega'$. So in this case $F(f) = 0$. As Ω' is arbitrary it may readily be determined by taking inductive limits that $F(f) = 0$ for each $f \in B^{p,l}_{\mathcal{C}\ell_{0,n}}(\Omega)$. This completes the proof. ∎

When $l = 1$ we can replace X by a countable dense subset of a hypersurface lying in the complement of the closure of Ω. See [10] for details.

Using the comments appearing immediately after Proposition 5 we can apply Proposition 6 to determine the following result.

Proposition 7. *Suppose that Ω is as in Proposition 6, $n/(n-l+1) < p < \infty$, and $g \in W^{p,3l-1-1/p}_{\mathcal{C}\ell_{0,n}}(\partial\Omega)$. Given the boundary value problem $\triangle^l u = 0$, in Ω and $tr_{\partial\Omega} u = g$ then provided $2l < n$ there exist Clifford numbers c_{i,n_i}, $i = 1, 2, \ldots, n_j$, $j = 1, 2, \ldots, l$, such that for each $\epsilon > 0$ and for the solution u of the boundary value problem, we have*

$$\left\| u - \sum_{j=1}^{2l} \sum_{i=1}^{n_j} \Psi^z_{i,j-1} c_{i,n_j} \right\|_{W^{p,k+2l}_{\mathcal{C}\ell_{0,n}}} < \epsilon.$$

5. Clifford analysis over hyperbolas

To begin our work, let us first start with the hyperbolas themselves. The hyperbolas $\mathbb{H}^n$ can be described as the set

$$\left\{ x = x_1 e_1 + \cdots + x_n e_n + x_{n+1} f_{n+1} \in R^{1,n} : -\sum_{j=1}^{n} {x_j}^2 + {x_{n+1}}^2 = 1 \right\}.$$

The hyperbolas defined above sitting in $R^{1,n}$ can also be viewed as the image of $\mathbb{R}^n \backslash S^{n-1}$ under the Cayley transformation

$$k \colon \mathbb{R}^n \backslash S^{n-1} \to \mathbb{H}^n \backslash \{-f_{n+1}\} \subset R^{1,n}$$

defined by

$$k(x) = (-x + f_{n+1})(f_{n+1} x + 1)^{-1}.$$

This transformation takes the open unit disk D_n onto the component $\mathbb{H}^n_+$, of the hyperbolas $\mathbb{H}^n$ which contains f_{n+1}. It also maps the complement of the closed unit disc to the component, $\mathbb{H}^n_-$, of $\mathbb{H}^n$ that contains $-f_{n+1}$. Thus $\mathbb{H}^n = \mathbb{H}^n_+ \cup \mathbb{H}^n_-$. A subset Ω_H of $\mathbb{H}^n$ is called a domain if $k^{-1}(\Omega_H)$ is a domain in $\mathbb{R}^n$ or if the union of a suitable subset of S^{n-1} with $k^{-1}(\Omega_H)$ is a domain in $\mathbb{R}^n$. See [18] for more details. So when working on general domains in $\mathbb{H}^n$, the domain may not have only one connected component, it may have one component in $\mathbb{H}^n_+$ and one in $\mathbb{H}^n_-$.

More details are given in [18]. In particular the following lemma is proved there.

Lemma 3. *Let k be the Cayley transformation defined above. Then*

$$E(u-v) = \tilde{J}(k,y)^{-1} E(x-y) J(k,x)^{-1}$$

where

$$J(k,x) = \frac{\widetilde{(f_{n+1}x + 1)}}{\| f_{n+1} x + 1 \|^n}$$

and $u = k(x)$, $v = k(y)$.

Definition 8. *A subset Ω_H of $\mathbb{H}^n$ is called a domain in $\mathbb{H}^n$ if there exists a domain U in $\mathbb{R}^n$ such that $k(U\backslash S^{n-1}) = \Omega_H$.*

Let $\Omega_H \subset \mathbb{H}^n$ be a domain and let f be a left-monogenic and g be a right-monogenic function in $k^{-1}(\Omega_H) \subset \mathbb{R}^n$. Then

$$0 = \int_\Sigma f(x)n(x)g(x)\,d\sigma_x = \int_{k(\Sigma)} f(k^{-1}(y))\tilde{J}(k^{-1},y)n(y)J(k^{-1},y)g(y)\,d\eta_y$$

where $y = k(x)$ and Σ is a Lipschitz surface inside $k^{-1}(\Omega_H)$ bounding some domain Ω and $n(y)$ is the unit vector lying in the tangent space $T\mathbb{H}^n_{k(x)}$ and is normal to the tangent space $Tk^{-1}(\Sigma)_{k(x)}$. Furthermore

$$J(k^{-1},y) = \frac{(\widetilde{f_{n+1}y} - 1)}{((f_{n+1}y-1)^2)^{n/2}},$$

and η is the boundary Lebesgue measure on $k(\Sigma)$. Using Stokes' Theorem, the right hand side of the previous integral equation will be

$$\int_{\Omega_H} \Big[(f(k^{-1}(y))\tilde{J}(k^{-1},y)D_{\mathbb{H}^n})J(k^{-1},y)g(k^{-1}(y)) \\ + f(k^{-1}(y))\tilde{J}(k^{-1},y)(D_{\mathbb{H}^n}J(k^{-1},y)g(k^{-1}(y)))\Big]\,d\lambda$$

where λ is the Lebesgue measure on $\mathbb{H}^n$. The symbol $D_{\mathbb{H}^n}$ is then the hyperbolic Dirac operator which arises from the application of Stokes' Theorem. Therefore, for the reason stated in [17, 18], we have that

$$\begin{aligned} f(k^{-1}(y))\tilde{J}(k^{-1},y)D_{\mathbb{H}^n} &= 0, \\ D_{\mathbb{H}^n}J(k^{-1},y)f(k^{-1}(y)) &= 0. \end{aligned}$$

Thus we have the following definition.

Definition 9. *Let $\Omega_{\mathbb{H}}$ be a domain in $\mathbb{H}^n$. Then a differentiable $\mathcal{C}\ell_{1,n}$ valued function f defined on Ω_H is said to be hyperbolic left-monogenic if*

$$D_{\mathbb{H}^n} f = 0$$

on Ω_H and a differentiable $\mathcal{C}\ell_{1,n}$-valued function g is called hyperbolic right-monogenic if $gD_{\mathbb{H}^n} = 0$

Example. The function

$$E(x-y) = \frac{1}{\omega_n}\frac{x-y}{((x-y)^2)^{n/2}}$$

is a hyperbolic left monogenic function and a hyperbolic right monogenic function.

The function

$$\Pi^w(x,y) = E(x-y) - E(x-w)$$

is also both left and right hyperbolic monogenic.

The following result is established in [17].

Theorem 2. *A function f is hyperbolic left monogenic in y if and only if $J(k^{-1},x)f(k^{-1}(x))$ is left monogenic in x and g is hyperbolic right monogenic in y if and only if $g(k^{-1}(x))\tilde{J}(k^{-1},x)$ is right monogenic in x.*

The following two results are obtained by adapting arguments given in $\mathbb{R}^n$ in [3, 4] and adapting Lemma 1 to the setting of the hyperbola.

Proposition 8 (Borel-Pompeiu)**.** *Let Ω_H be an unbounded domain with piecewise C^1 boundary and $g\colon \Omega_H \to \mathcal{C}\ell_{1,n}$ is a C^1 function with bounded continuous extension to the boundary and $D_{\mathbb{H}^n}g$ is in $L^p_{\mathcal{C}\ell_{1,n}}(\Omega_H)$ for some $p \in (1,\infty)$. Suppose also that Ω_H has an open set in the closure of its complement and w belongs to this open set. Then for every $y \in \Omega_H$ we have*

$$g(y) = \int_{\partial\Omega_H} \Pi^w(x,y)n(x)g(x)\,d\sigma(x) + \int_{\Omega_H} \Pi^w(x,y)D_{\mathbb{H}^n}g(x)\,d\eta(x).$$

Corollary 2 (Cauchy-Integral-Formula)**.** *Suppose Ω_H is the same as in the previous proposition. If g is a bounded hyperbolic left-monogenic on Ω_H then*

$$g(y) = \int_{\partial\Omega_H} \Pi^w(x,y)n(x)g(x)\,d\eta(x).$$

Definition 10. *Let Ω_H be a domain in $\mathbb{H}^n$ and $1 < p < \infty$. Then we define the p-Bergman space of hyperbolic left monogenic functions over Ω_H by* $\ker D_{\mathbb{H}^n}(\Omega_H) \cap L^p_{\mathcal{C}\ell_{1,n}}(\Omega_H)$ *and it is denoted by $B^p_{\mathcal{C}\ell_{1,n}}(\Omega_H)$.*

By very similar techniques to those used to establish Theorem 1 one can establish the following result.

Theorem 3. *For $1 < p < \infty$ the space $B^p_{\mathcal{C}\ell_{1,n}}$ is complete.*

Proposition 9. *Suppose Ω_H is an unbounded domain in $\mathbb{H}^n$. Then*

$$L^p_{\mathcal{C}\ell_{1,n}}(\Omega_H) = B^p_{\mathcal{C}\ell_{1,n}}(\Omega_H) \oplus (D_{\mathbb{H}^n} + x)(W^{0,p,1}_{\mathcal{C}\ell_{1,n}}(\Omega_H))$$

where $L^p_{\mathcal{C}\ell_{1,n}}(\Omega_H)$ is the space of p-integrable functions over $\Omega_H(\mathcal{C}\ell_{1,n})$ and the space $W^{0,p,1}_{\mathcal{C}\ell_{1,n}}(\Omega_H)$ is the L^p-completion of smooth functions on Ω_H with compact support.

Remark. The second order hyperbolic Dirac operator or hyperbolic Laplacian, $\triangle_{\mathbb{H}^n}$ or $D^{(2)}_{\mathbb{H}^n}$, is shown in [11] to be

$$\triangle_{\mathbb{H}_n} := (D_{\mathbb{H}^n} - x)D_{\mathbb{H}^n} = D_{\mathbb{H}^n}(D_{\mathbb{H}^n} + x) \qquad \text{for } x \in \mathbb{H}^n.$$

Proof of Proposition 9. Let $f \in L^p_{\mathcal{C}\ell_{1,n}}(\Omega_H)$ and consider the function

$$h = \triangle^{-1}_{\mathbb{H}^n,0} D_{\mathbb{H}^n} f \in W^{0,p,1}_{\mathcal{C}\ell_{1,n}}(\Omega_H),$$

where $\triangle_{\mathbb{H}^n,0}$ is the restriction of $\triangle_{\mathbb{H}^n}$ to $W^{0,p,1}_{\mathcal{C}\ell_{1,n}}(\Omega_H)$. Let

$$g = (D_{\mathbb{H}^n} + x)h \in (D_{\mathbb{H}^n} + x)(W^{0,p,1}_{\mathcal{C}\ell_{1,n}}(\Omega_H))$$

and $\psi = f - g$. We have $\psi \in B^p_{\mathcal{C}\ell_{1,n}}(\Omega_H)$ and so $f = \psi + g$.

Also, suppose f is in both the two summands, then $D_{\mathbb{H}^n} f = 0$ and $f = (D_{\mathbb{H}^n}+x)g$ for some $g \in W^{0,p,1}_{\mathcal{C}\ell_{1,n}}(\Omega_H)$, which implies $D_{\mathbb{H}^n}(D_{\mathbb{H}^n} + x)g = D_{\mathbb{H}^n} f = 0$. But since $\ker \triangle_{\mathbb{H}^n,0} = \{0\}$, we have that $f \equiv 0$. ■

This decomposition gives projections

$$\begin{aligned} P_H &: L^p_{\mathcal{C}\ell_{1,n}}(\Omega_H) \to B^p_{\mathcal{C}\ell_{1,n}}(\Omega_H), \\ Q_H &: L^p_{\mathcal{C}\ell_{1,n}}(\Omega_H) \to (D_{\mathbb{H}^n} + x)(W^{0,p,1}(\Omega_H)) \end{aligned}$$

with $Q_H = I_H - P_H$, where I_H is the identity operator over $\mathbb{H}^n$.

As we also have the factorization

$$\triangle_{\mathbb{H}^n} = (D_{\mathbb{H}^n} - x)D_{\mathbb{H}^n},$$

we are also interested in solutions to the equation $(D_{\mathbb{H}^n} - x)f = 0$. In [12] it is shown that the generalized Cauchy kernel for the Dirac operator $D_{\mathbb{H}^n} - x$ for bounded domains is the function

$$Q(x,y) = \frac{x-y}{((x-y)^2)^{n/2}} + x\frac{1}{((x-y)^2)^{n/2}}.$$

For an unbounded domain Ω_H, whose complement contains an open set the kernel $Q(x,y)$ may be modified to

$$\Theta^w(x,y) = \frac{1}{\omega_n}(Q(x,y) - Q(x,w)),$$

where, as before, w lies in the complement of the closure of Ω_H.

Definition 11. *A differentiable function $f\colon \Omega_H \to \mathcal{C}\ell_{1,n}$ is called an alternative left hyperbolic monogenic function if $(D_{\mathbb{H}^n} + x)f(x) = 0$ for each $x \in \Omega_H$.*

A similar definition can be given for alternative right hyperbolic monogenic functions.

Arguments similiar to those used to prove Proposition 1 show the following result.

Proposition 10. *The kernel $\Theta^w(x,y)$ belongs to $L^p_{\mathcal{C}\ell_{1,n}}(\Omega_{H,\epsilon})$ for $n/(n-1) < p < \infty$, where $\Omega_{H,\epsilon} = \Omega_H \backslash B(y,\epsilon)$ and $B(y,\epsilon) = \{x \in \mathbb{H}^n : \|x-y\| < \epsilon\}$ and $\|x\|$ is the norm of $x \in R^{1,n}$ when $R^{1,n}$ is identified with $\mathbb{R}^{n+1}$ in the usual way.*

Definition 12. *Suppose Ω_H is a domain in $\mathbb{H}^n$ and $n/(n-1) < p < \infty$. Then the space* $\ker(D_{\mathbb{H}^n} - x)(\Omega_H) \cap L^p_{\mathcal{C}\ell_{1,n}}(\Omega_H)$ *is defined to be the p-th Bergman space $C^p_{\mathcal{C}\ell_{1,n}}(\Omega_H)$.*

By similar arguments to those used to establish Proposition 9 we may deduce the following result.

Proposition 11. *Suppose* Ω_H *is a domain in* $\mathbb{H}^n$ *with* C^2 *boundary. Then*

$$L^p_{\mathcal{C}\ell_{1,n}}(\Omega_H) = C^p_{\mathcal{C}\ell_{1,n}}(\Omega_H) \oplus (D_{\mathbb{H}^n} - x)W^{0,p,1}_{\mathcal{C}\ell_{1,n}}(\Omega_H).$$

The next result gives us the invertibility of the hyperbolic Dirac operator acting over appropriate function spaces.

Proposition 12. *Let* Ω_H *be a* C^1 *domain in* $\mathbb{H}^n$*, and let* $f \in L^p_{\mathcal{C}\ell_{1,n}}(\Omega_{H^n})$*. Then*

$$D_{\mathbb{H}^n} \int_{\Omega_H} \Pi^w(u,v) f(u)\, d\lambda = f(v)$$

for each $v \in \Omega_H$*. Thus the transform*

$$T^\star_{\Omega_H} f(v) := \int_{\Omega_H} \Pi^w(u,v) f(u)\, d\lambda$$

is the right inverse of the hyperbolic Dirac operator $D_{\mathbb{H}^n}$*.*

It would be nice to have an analogue of Proposition 3 in the context of the hyperbola. To do this one needs to determine an inverse for the operator $D_{\mathbb{H}^n} - x$ in a similar context similar to that described in the previous proposition. Such a kernel has not yet been found.

By the same arguments to those used in [6] to establish a version of Lusin's Theorem for left monogenic functions in Euclidean space we can establish the following version of Lusin's Theorem for hyperbolic left monogenic functions.

Theorem 4. *Suppose that* $f\colon \Omega_H \to \mathcal{C}\ell_{1,n}$ *is a hyperbolic left monogenic function and for some* C^2 *hypersurface* $\Sigma \subset \Omega_H$ *the function* f *vanishes on* Σ*. Then* $f \equiv 0$*.*

Using this theorem we can establish an analogue for hyperbolas of a result proved in [10] on complete function spaces for unbounded domains in Euclidean space. As the proof follows the same lines to the one given in [10] we shall omit it.

Proposition 13. *Let* Ω_H *be a unbounded domain in* $\mathbb{H}^n$ *which satifies the conditions of Proposition 9. Let* $X = \{u_m : m \in \mathbb{N}\}$ *be a dense subset of some smooth hypersurface* Σ *lying in the complement of the closure of* Ω_H*. Suppose furthermore that this hypersurface is homologous within* $\mathbb{H}^n$ *to* $\partial\Omega_H$*. Also suppose* w *lies in the complement of the closure of* Ω_H *and* $w \notin \Sigma$*. Then the function system* $\{\Pi^w(x_m, y) : m \in \mathbb{N}\}$ *is* $\mathcal{C}\ell_{1,n}$*-complete in* $B^p_{\mathcal{C}\ell_{1,n}}(\Omega_H)$ *for* $1 < p < \infty$*.*

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