ON SINGULAR INTEGRAL OPERATORS

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Abstract. In this paper we study singular integral operators which are hyper or weak over Lipscitz/Hölder spaces and over weighted Sobolev spaces defined on unbounded smooth domains in the standard n-D Euclidean space $\mathbb{R}^n$, where $n \geq 1$. The $\pi-$operator in this case is one of the hypersingular integral operators which has been studied extensibly than other hyper singular integral operators. It will be shown the control of singularity of hyper singular integral operators that are defined interms of Cauchy generating kernels by working on weighted function spaces such as $W^{p,k}\left(\Omega, \| x \|^{\zeta+\epsilon}\, dx\right)$ for some $\epsilon > 0$ and ζ , some positive integer. The latter spaces usually are termed as weighted Sobolev spaces.

1. **Singular Integral Operators**

In this short note we discuss few points about super singular integral operators, weak(or sub) singular and just singular integral operators by showing few examples and present some results.

We therefore introduce general singular integral operators in terms of integrals with Cauchy generating kernels and some other general singular integral operators with out kernels.

The calculus versions of singular integral operators are improper integrals, integrals with unbounded integrands or integrals with unbounded intervals of integrations.

To start our work, let Ω be some bounded domain in the Euclidean space $\mathbb{R}^n$ and ψ be some integrable function over Ω and $x_0 \in \Omega^{\text{int}}$,

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Date: August 19, 2008.

2000 *Mathematics Subject Classification.* Primary 30G35,35A22.

Key words and phrases. Hyper singular operaors, Cauchy kernels, weighted Sobolev spaces.

This paper is in final form and no version of it will be submitted for publication elsewhere.

interior of the domain with the property that

$$\lim_{x \to x_0} | \psi(x) |= \infty$$

which in this case x_0 is a singular point of the function.

The integral given by

$$\int_{\Omega} \psi(x)\, dx$$

is called a singular integral of the function ψ over the domain Ω with a singularity point x_0.

We evaluate such singular integrals by evaluating the Cauchy principal value of the singular integral which is given as follows.

Let $\epsilon > 0$ and consider the ball $B(x_0, \epsilon)$ and define $\Omega_\epsilon := \Omega \backslash B(x_0, \epsilon)$. Then we consider the integral over the deleted sub domain Ω_ϵ by

$$\int_{\Omega_\epsilon} \psi(x)\, dx$$

which avoids the singularity x_0.

If the limit :

$$\lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \psi(x)\, dx$$

called the Cauchy principal value(c.p.v.) exits, then we define the value of the singular integral as:

$$\int_{\Omega} \psi(x)\, dx := \lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \psi(x)\, dx$$

Examples of elementary singular integral operators are given below:

In the unidimensional Euclidean space $\mathbb{R}^1$: let $\Omega = (-1, 1)$ and define the function by

$$\psi_\alpha(x) =| x |^{-\alpha}, \quad \text{for} \quad 0 < \alpha < 1$$

Then the function ψ_α has a singularity at 0, since

$$\lim_{x \to 0} | \psi_\alpha(x) |= \infty$$

Therefore, the integral given by $\int_{\Omega} \psi_\alpha(x)\, dx$ is a weakly singular integral

Let $\epsilon > 0$ and consider

$$\Omega_\epsilon = \Omega \backslash B(0,\epsilon) = (-1,1) \setminus (-\epsilon,\epsilon).$$

Then the integral $\int\limits_{\Omega_\epsilon} \psi_\alpha(x)\,dx$ is no more a singular integral at x_0 and therefore has a finite integral as long as the function ψ is integrable on the domain Ω.

Therefore,

$$\int\limits_{\Omega_\epsilon} \psi_\alpha(x)\,dx = \int\limits_{\Omega_\epsilon} |\, x \,|^{-\alpha}\,dx$$

is a function of α and ϵ and if we denote this function by $I(\alpha,\epsilon)$, then we have

$$I(\alpha,\epsilon) = \frac{2}{1-\alpha}\left(1-\epsilon^{1-\alpha}\right)$$

which is a finite value in terms of ϵ and α. Then taking the c.p.v. of the above integral :

$$\begin{aligned} \lim_{\epsilon\to 0} \int\limits_{\Omega_\epsilon} \psi_\alpha(x)\,dx &= \lim_{\epsilon\to 0} \int\limits_{\Omega_\epsilon} |\, x \,|^{-\alpha}\,dx \\ &= \lim_{\epsilon\to 0} I(\alpha,\epsilon) \\ &= \frac{2}{1-\alpha} \end{aligned}$$

as $1-\alpha > 0$.

When $\alpha = 1$, the function is $\psi_{-1}(x) = |\, x \,|^{-1}$ and this function generates an integral $\int\limits_{\Omega} \psi_{-1}(x)\,dx$ called a singular integral.

For $\alpha = 1+\varepsilon$, $\varepsilon > 0$, the integral $\int\limits_{\Omega} \psi_\alpha(x)\,dx$ is called a hyper singular integral. Besides

$$\begin{aligned} \lim_{\epsilon\to 0} \int\limits_{\Omega_\epsilon} \psi_\alpha(x)\,dx &= \lim_{\epsilon\to 0} \int\limits_{\Omega_\epsilon} |\, x \,|^{-\alpha}\,dx \\ &= \lim_{\epsilon\to 0} I(\alpha,\epsilon) \\ &= \lim_{\epsilon\to 0} \frac{2}{1-\alpha}\left(1-\epsilon^{1-\alpha}\right) = \infty \end{aligned}$$

Therefore the improper integral is divergent.

We therefore construct the classical singular integral operators which are obtained from generating Kernels.

Let us begin with one of the most common generating kernels given by the function:

$$K(x) = \frac{\overline{x}}{\omega_n \mid x \mid^n}$$

which is called the Cauchy kernel whose singularity is at zero.

This kernel gives singular integral operator on the space of functions such that the convolution is finite over the domain Ω, which is given by

$$\Phi(\psi)(x) = \int_{\Omega} K(x-y)\,\psi(y)\,d\Omega_y$$

From the classification of singular integrals, we will see that Φ is indeed a weak singular integral: let $\lambda \in \mathbb{R}_{>0}$,

$$K(\lambda x) = \frac{\lambda \overline{x}}{\omega_n \lambda^n \mid x \mid^n} = \lambda^{-(n-1)} K(x)$$

which gives that K is a homogeneous function of exponent $n-1$ which is less than n.

The singular integral operator Φ given above in literature is called the Teodorescu transform.

It is an important transform in Sobolev spaces with a regularity augmentation property by one :

$$\Phi : W^{p,k}(\Omega) \to W^{p,k+1}(\Omega).$$

We can further study the function spaces where the weak singular integral works. In the sequel, we use the following set up:

For $\varepsilon > 0$, consider $B(x,\varepsilon)$, the $\varepsilon-$ball centered at x and radius ε and consider the punctured domain $\Omega_\varepsilon = \Omega \backslash B(x,\varepsilon)$.

Proposition 1. *If Ω is unbounded and smooth domain in $\mathbb{R}^n$, then $K(x)$ is $p-$integrable over Ω_ε for $\frac{n}{n-1} < p < \infty$.*

Proof.

$$\|K(x)\| = \|\frac{\overline{x}}{\omega_n \mid x \mid^n}\| = \frac{r^{1-n}}{\omega_n}$$

for $\|x\| = r$ and using polar coordinates, we have the following norm estimates:

$$\int_{\Omega_\varepsilon} \|K(x)\|^p dx \leq c(\theta) \int_{\varepsilon}^{\infty} r^{p(1-n)+n-1} dr$$

$$= c(\theta) \lim_{\sigma\to\infty} \left(\frac{r^{p(1-n)+n}}{p(1-n)+n} \Big|_{\varepsilon}^{\sigma} \right)$$

$$= c(\theta) \lim_{\sigma\to\infty} \left(\frac{\sigma^{p(1-n)+n}}{p(1-n)+n} - \frac{\varepsilon^{p(1-n)+n}}{p(1-n)+n} \right)$$

and this is finite and equals $c(\theta)\left(\frac{\varepsilon^{p(1-n)+n}}{p(n-1)-n}\right)$, if

$$p(1-n)+n < 0$$

That is

$$\frac{n}{n-1} < p < \infty$$

which proves the proposition. □

If the domain is a bounded smooth one, then we consider a singularity at a finite point and the exponent of integrability will be different.

Now, as we see that K is in the Sobolev space $W^{p,k}(\Omega_\varepsilon)$ for $p > \frac{n}{n-1}$, we can determine the function space where we can work with this function as a generating kernel for singular integral operators .

Proposition 2. *The convolution $K* |_{\Omega_\varepsilon} f$ is well defined and finite over $W^{q,k}(\Omega_\varepsilon)$ for $1 < q < n$.*

Proof. From Hölder's inequality, the product $Kf \in W^{1,k}(\Omega_\varepsilon)$ when $K \in W^{p,k}(\Omega_\varepsilon)$ and $f \in W^{q,k}(\Omega_\varepsilon)$ such that $p^{-1} + q^{-1} = 1$.

Therefore as $p \in (\frac{n}{n-1}, \infty)$, we have $q \in (1, n)$ which is the required result. □

Corollary 1. *When Ω is a $2-D$ domain, the Sobolev index p should strictly be greater than 2. Therefore the generating kernel does not work over $W^{2,k}(\Omega)$.*

Proposition 3. *Let Ω be a smooth, unbounded domain in $\mathbb{R}^n$ and $p \in (\frac{n}{n-1}, \infty)$ and q be the conjugate index of p. Then we have :*

$$\| Kf \|_{W^{1,k}(\Omega_\epsilon)} \leq \| K \|_{W^{p,k}(\Omega_\varepsilon)} \| f \|_{W^{q,k}(\Omega_\varepsilon)} \rightarrow \| K \|_{W^{p,k}(\Omega_\varepsilon)} \| f \|_{W^{n,k}(\Omega_\varepsilon)}$$

as $q \nearrow n$.

Proof. From Hölder's inequality, we have

$$\| Kf \|_{W^{1,k}(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} | Kf | \leq \| K \|_{W^{p,k}(\Omega_\varepsilon)} \cdot \| f \|_{W^{q,k}(\Omega_\varepsilon)}$$

Then taking the limiting norm on the indices p and q with $p^{-1}+q^{-1}=1$ we have :

$$\lim_{q \nearrow n} \left(\| K \|_{W^{p,k}(\Omega_\varepsilon)} \cdot \| f \|_{W^{q,k}(\Omega_\varepsilon)} \right) = \| K \|_{W^{p,k}(\Omega_\varepsilon)} \cdot \| f \|_{W^{n,k}(\Omega_\varepsilon)}$$

since $p \searrow \left(\frac{n}{n-1}\right) \Rightarrow q \nearrow n$ and that finishes the argument. □

The next singular integral we consider is the one generated from the fundamental solution of the Laplacian operator

$$\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} = \left(\sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j} \right) \overline{\left(\sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j} \right)}$$

which is given by $\Psi_{2.\Omega}(x) = \frac{-1}{\omega_n \|x\|^{n+2}}$ and the corresponding singular integral associated is given by

$$\Phi_{2,\Omega}(\phi) = \int_{\Omega} \Psi_{2.\Omega}(x-y) \, \phi(y) \, d\Omega_y$$

We investigate in which generalized Lebesgue space is $\Psi_{2.\Omega}$ over unbounded domain $\Omega \subseteq \mathbb{R}^n$.

Proposition 4. *Let Ω be a smooth and unbounded domain in $\mathbb{R}^n$ for $n \geq 1$. Then $\Psi_{2.\Omega} \in W^{p,k}(\Omega_\varepsilon, Cl_n)$ for $p \in (\frac{n}{n+2}, \infty)$.*

Proof. Consider the integral $\int_{\Omega_\delta} | \Psi_{2.\Omega} |^p \, dx$, using polar coordinates, the integral becomes : $c(\theta, \omega_n) \int_{\delta}^{\infty} r^{-(n+2)p+n-1} dr$ and it will be finite towards the boundary of the domain when $p > \frac{n}{n+2}$, where $c(\theta, \omega_n)$ is a constant that depends on θ and the surface area ω_n of the unit sphere S^{n-1}. □

2. **Weighted Sobolev Spaces**

If we try to find Sobolev spaces in which the kernel $\Psi_{2.\Omega}$ works, we might end up in working with a dual spaces whose conjugate indices are negative.

For instance in the limiting cases : $q \longrightarrow \frac{-n}{2}$ as $p \searrow \frac{n}{n+2}$, which shows that q has a negative limiting index which is going to be a conjugate index of a limiting index of p in some sense.

To remedy this, we introduce a weight on the Lebesgue volume measure dx so that we avoid dual spaces with negative indices.

The weight function that we choose stretches the Lebesgue volume measure so that the singularity from the kernel is better managed and made more controlled.

We choose a radial weight function given by $w(x) = \parallel x \parallel^{2+\varepsilon}$, where ε is some positive constant and we investigate the integral :

$$\int\limits_{\Omega_\delta} \Psi_{2.\Omega}(x) d\mu(x)$$

where $d\mu(x) = w(x)dx$.

Proposition 5. *Over unbounded domain* $\Omega \subseteq \mathbb{R}^n$, $\Psi_{2.\Omega} \in W^{p,k}(\Omega_\delta)$ *for* $1 + \frac{\varepsilon}{n+2} < p < \infty$.

Proof. We see from the proposition that the interval for the index p is much improved and the conjugate space will be a dual space with positive index.

Therefore,

$$\int\limits_{\Omega_\delta} \| \quad \Psi_{2.\Omega} \|^p \, d\mu(x) = c(\theta, \omega_n) \int\limits_{\delta}^{\infty} r^{-(n+2)p+2+n+\varepsilon-1} dr$$

$$= \quad c(\theta, \omega_n) \lim_{\rho \to \infty} \left(\frac{r^{-(n+2)p+n+2+\varepsilon}}{-(n+2)p+n+2+\varepsilon} \, |_{\delta}^{\rho} \right) < \infty$$

when $-(n+2)p+n+2+\varepsilon < 0$ which implies that $p > \frac{2+n+\varepsilon}{2+n}$ which is the required result. □

Next, we determine the Sobolev space $W^{q,k}(\Omega)$ in which the product $\Psi_{2.\Omega}\phi$ is integrable or the convolution $\Psi_{2.\Omega} *_{|\Omega} \phi$ is finite.

Proposition 6. *Over unbounded domain $\Omega \subseteq \mathbb{R}^n$, and for $1 + \frac{\varepsilon}{n+2} < p < \infty$, with respect to the weighted measure $d\mu(x) = \| x \|^{2+\varepsilon} dx$ we have*

$\Psi_{2.\Omega}\phi \in W^{1,k}(\Omega, \| x \|^{2+\varepsilon} dx)$ when $\phi \in W^{q,k}(\Omega, \| x \|^{2+\varepsilon} dx)$ for $1 < q < 1 + \frac{n+2}{\varepsilon}$.

Proof. From the previous proposition, for $p > \frac{2+n+\varepsilon}{2+n}$, we proved that $\Psi_{2.\Omega} \in W^{p,k}(\Omega, \| x \|^{2+\varepsilon} dx)$.

Therefore, if ϕ is a function in $W^{q,k}(\Omega, \| x \|^{2+\varepsilon} dx)$ such that $p^{-1} + q^{-1} = 1$, we have the integral estimate :

$$\int\limits_{\Omega} | \, \Psi_{2.\Omega}\phi \, | \leq \| \, \Psi_{2.\Omega} \, \|_{W^{p,k}(\Omega, \|x\|^{2+\varepsilon} dx)} \cdot \| \, \phi \, \|_{W^{q,k}(\Omega, \|x\|^{2+\varepsilon} dx)}$$

where $1 < q < 1 + \frac{n+2}{\varepsilon}$. □

Proposition 7. *Let $\Omega \subseteq \mathbb{R}^n$ be a unbounded smooth domain , and $1 + \frac{\varepsilon}{n+2} < p < \infty$, then we have the following norm estimates:*

$$\begin{aligned} & \| \quad \Psi_{2.\Omega} \|_{W^{p,k}(\Omega, \|x\|^{2+\varepsilon} dx)} \cdot \| \, \phi \, \|_{W^{\left(1+\frac{n+2}{\varepsilon}\right),k}(\Omega, \|x\|^{2+\varepsilon} dx)} \\ \leq & \quad \| \, \Psi_{2.\Omega} \, \|_{W^{\left(1+\frac{\varepsilon}{n+2}\right),k}(\Omega, \|x\|^{2+\varepsilon} dx)} \cdot \| \, \phi \, \|_{W^{q,k}(\Omega, \|x\|^{2+\varepsilon} dx)} \end{aligned}$$

and

$$\begin{aligned} & \lim_{q \nearrow \left(1+\frac{n+2}{\varepsilon}\right)} \left(\| \Psi_{2.\Omega} \|_{W^{p,k}(\Omega, \|x\|^{2+\varepsilon} dx)} \cdot \| \phi \|_{W^{q,k}(\Omega, \|x\|^{2+\varepsilon} dx)} \right) \\ = \; & \| \Psi_{2.\Omega} \|_{W^{p,k}(\Omega, \|x\|^{2+\varepsilon} dx)} \cdot \| \phi \|_{W^{\left(1+\frac{n+2}{\varepsilon}\right),k}(\Omega, \|x\|^{2+\varepsilon} dx)} \end{aligned}$$

Proof. Note that the Sobolev norm used here is with respect to th weighted measure $d\mu(x) = \| x \|^{2+\varepsilon} \, dx$. The first part of the proposition follows from the decreasing monotonic nature of Lebesgue norm with respect to the increase in the index since $q \nearrow_1^{\left(1+\frac{n+2}{\varepsilon}\right)}$ and the second follows from the general theory of continuity of Lebesgue norm. ☐

Corollary 2. *When $n = 2$, we have :*

$$\begin{aligned} & \| \Psi_{2.\Omega} \|_{W^{,pk}(\Omega, \|x\|^{2+\varepsilon} dx)} \cdot \| \phi \|_{W^{\left(1+\frac{4}{\varepsilon}\right),k}(\Omega, \|x\|^{2+\varepsilon} dx)} \\ \leq \; & \| \Psi_{2.\Omega} \|_{W^{\left(1+\frac{\varepsilon}{4}\right),k}(\Omega, \|x\|^{2+\varepsilon} dx)} \cdot \| \phi \|_{W^{q,k}(\Omega, \|x\|^{2+\varepsilon} dx)} \end{aligned}$$

and

$$\begin{aligned} & \lim_{q \nearrow \left(1+\frac{4}{\varepsilon}\right)} \left(\| \Psi_{2.\Omega} \|_{W^{p,k}(\Omega, \|x\|^{2+\varepsilon} dx)} \cdot \| \phi \|_{W^{q,k}(\Omega, \|x\|^{2+\varepsilon} dx)} \right) \\ = \; & \| \Psi_{2.\Omega} \|_{W^{p,k}(\Omega, \|x\|^{2+\varepsilon} dx)} \cdot \| \phi \|_{W^{\left(1+\frac{4}{\varepsilon}\right),k}(\Omega, \|x\|^{2+\varepsilon} dx)} \end{aligned}$$

3. **Generating kernels:** $\Psi_{l,\Omega}(x)$

In this section, we extrapolate the idea of constructing singular integral operators as convolutions with fundamental solutions of the Dirac operator to the once generated by fundamental solutions of higher iterates of the Dirac operator.

Some kernels generate hyper singular integral operators and others form weaker singular integral operators. It is therefore interesting to look at differences of these formations from the very constructions of the operators.

These functions are constructed by recursive (or iterative) way from the fundamental solutions of the Dirac operator and its higher iterates and are given below:

$$\Psi_{l,\Omega}(x) = \begin{cases} \theta(n,l)\frac{x}{\omega_n\|x\|^{n-l+1}}, \text{ if } l \text{ is odd} \\ \\ \frac{\theta(n,l)}{\omega_n\|x\|^{n-l+1}}, \text{if } l \text{ is even} \end{cases}$$

where $l < n$.

For a detail study of the constructions of these functions and their application for constructing complete family of functions and minimal family of functions, one can see [5],[6]

Proposition 8. *For $l < n$ and $\Omega^{unbdd,\ smooth} \subseteq \mathbb{R}^n$, the function $\Psi_{l,\Omega} \in W^{p,k}(\Omega_\varepsilon, Cl_n)$ for*
$$\begin{cases} \frac{n}{n-l} < p < \infty, \text{ when } l \text{ is odd} \\ \\ \frac{n}{n+1-l} < p < \infty, \text{ when } l \text{ is even.} \end{cases}.$$

Proof. For Ω unbounded and smooth with $\Omega_\varepsilon = \Omega \backslash B(x,\varepsilon)$ for $\varepsilon > 0$, using polar coordinates, the integral

$$\int_{\Omega_\varepsilon} \|\Psi_{l,\Omega}(x)\|^p dx$$

is dominated by the integral

$$C(\theta, n, \omega_n) \int_\varepsilon^\infty r^{-p(n-l)+n-1} dr$$

for l odd with finite integral when the index p satisfies the inequality

$$\frac{n}{n-l} < p < \infty$$

and when l is even, it is dominated by the integral:

$$C(\theta, n, \omega_n) \int_\varepsilon^\infty r^{-p(n+1-l)+n-1} dr$$

which again is convergent for the indices which satisfy the inequality:

$$\frac{n}{n+1-l} < p < \infty$$

where $C(\theta, n, \omega_n)$ is some constant that depends on n, θ and ω_n. □

Thus for l : odd, when we work with this generating kernels, we have the indices p that depends on l and n and the conjugate index q has the following limiting values: as $p \rightarrow \frac{n}{n-l}$, we have : $q \rightarrow \frac{n}{l}$.

Thus, as $\Psi_{l,\Omega} \in W^{p,k}(\Omega, Cl_n)$ for $\frac{n}{n-l} < p < \infty$, the working Sobolev spaces for these kernels are $W^{q,k}(\Omega, Cl_n)$ for $1 < q < \frac{n}{l}$ such that $p^{-1} + q^{-1} = 1$.

Therefore for $\phi \in W^{q,k}(\Omega, Cl_n)$, we have the convergence of the *sub-singular* or in the literature terminology *weak* singular integral operators :

$$\int_{\Omega_\varepsilon} \Psi_{l,\Omega}(x)\,\phi(x)\,dx$$

with the usual integral inequality:

$$\left(\int_{\Omega_\varepsilon} \|\Psi_{l,\Omega}(x)\,\phi(x)\,\|dx\right)^{pq} \leq \int_{\Omega_\varepsilon} \|\Psi_{l,\Omega}(\ x)\,\|^p dx \int_{\Omega_\varepsilon} \|\phi(\ x)\,\|^q dx$$

For l even, we have the conjugate index $q \to \frac{n}{l-1}$ as $p \downarrow \frac{n}{n+1-l}$ and since $l < n$, we have that $\frac{n}{l-1} > 1$ and therefore, the above inequality holds again.

Then as convolution, we have :

Proposition 9. *For* $1 < q < \frac{n}{l}$*,or* $1 < q < \frac{n}{l-1}$ *, the integral operator* $\int_{\Omega} \Psi_{l,\Omega}(x-y)\,\phi(y)\,dy$ *is a weak-singular integral operator from*

$$W^{q,k}(\Omega, Cl_n) \to W^{q,k+1}(\Omega, Cl_n).$$

Proof. First, as $\Psi_{l,\Omega} \in W^{p,k}(\Omega, Cl_n)$ for $1 < p < \infty$, we have that for $\phi \in W^{q,k}(\Omega, Cl_n)$, for $1 < q < \frac{n}{l}$ (for l odd) or for $1 < q < \frac{n}{l-1}$ (for l even) with $p^{-1} + q^{-1} = 1$

such that the integral $\int_{\Omega_\varepsilon} \Psi_{l,\Omega}(x-y)\,\phi(y)\,dy$ is convergent but singular with out the puncture . The convolution is the usual Teodorescu transform which has the mapping property :

$$\Psi_{l,\Omega} * \phi : W^{q,k}(\Omega, Cl_n) \to W^{q,k+1}(\Omega, Cl_n).$$

□

Proposition 10. *In the usual* $3 - D$ *Euclidean space, if* $l = 3$*, then we can not work on the usual generalized Hilbert space* $W^{2,k}(\Omega, Cl_n)$*.*

Proof. For such a setting, we have that $3 < p < \infty$ and therefore the working function spaces will have conjugate Sobolev indices with range $1 < q < \frac{3}{2}$, in which the index 2 is not included.

Therefore the Sobolev space of index 2 which is the generalized Hilbert space $W^{2,k}(\Omega, Cl_n)$ is no more a viable space. □

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