On The Dirac Delta Generalized Function

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The Dirac delta generalized function δ is described intutively as a distribution with the following properties:

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

$$\int_{\mathbb{R}} \delta(x)\, dx = 1$$

and

$$\int_{\mathbb{R}} \delta(x)\, v(x)\, dx = v(0)\, , \forall v \in C_0^{\infty}(\mathbb{R})$$

We can also see $\delta(x)$ as the a distributional derivative of a function called Heavyside function given by:

$$h(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

by looking at the equality :

$$\int \delta(x)\, v(x) dx = -\int h(x) v'(x) dx$$

Indeed,

$$\begin{aligned} \int h(x) v'(x) dx &= \int_0^{\infty} v'(x) \\ &= -v(0) \end{aligned}$$

and so

$$-\int h(x) v'(x) dx = v(0)$$

On the other hand

$$\int_{\mathbb{R}} \delta(x)\, v(x) dx = v(0), \forall v \in C_0^{\infty}(\mathbb{R})$$

Therefore we conclude that

$$h'(x) = \delta(x)$$

in a sense of distributional or some times called weak derivative.

By taking distributional derivatives of h of all orders, we can see that the Dirac delta distribution is differentibale infinitely many times as follows:

First

$$\delta'(x) = h''(x)$$

as a distribution with:

$$\begin{aligned}\int_{\mathbb{R}} h''(x)\, v(x) dx &= \int_{\mathbb{R}} \delta'(x)\, v(x) dx \\ &= \int_{\mathbb{R}} h(x) v''(x) dx\end{aligned}$$

Indeed

$$\begin{aligned}\int_{\mathbb{R}} h(x) v''(x) dx &= \int_0^\infty v''(x) dx \\ &= v'(x) \mid_0^\infty \\ &= -v'(0)\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}} \delta'(x)\, v(x) dx &= -\int_{\mathbb{R}} \delta(x)\, v'(x) dx \\ &= -v'(0)\end{aligned}$$

which justifies δ' exists as a distribution. Then for an arbitrary $k \in \mathbb{N}$, we claim that

$$\delta^{(k)} = h^{(k+1)}$$

the $k-th$ distributional derivative of δ exists from:

$$\begin{aligned}\int_{\mathbb{R}} \delta^{(k)}(x)\, v(x) dx &= (-1)^k \int_{\mathbb{R}} \delta(x)\, v^{(k)}(x) dx \\ &= (-1)^k v^{(k)}(0), \forall v \in C_0^\infty(\mathbb{R})\end{aligned}$$

Therefore δ is infinitely differentiable generalized function and because there is no a regular function that behaves as δ does, it is called generalized function or distribution.

Definition 1 *The operator* $e^{\frac{d}{dx}}$ *is defined as a differential operator of infinite order by :* $e^{\frac{d}{dx}} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{dx^k}$

Proposition 2 $e^{\frac{d}{dx}}(e^x) = e^{(x+1)}$

Proof. Clearly e^x is an infitely differentiable function and therefore,

$$\begin{aligned} e^{\frac{d}{dx}}\left(e^x\right) &= \sum_{k=0}^{\infty}\frac{1}{k!}\frac{d^k e^x}{dx^k} \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}e^x \\ &= e^x\sum_{k=0}^{\infty}\frac{1}{k!} \\ &= e^{x+1} \end{aligned}$$

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Proposition 3 *$e^{\frac{d}{dx}}\delta = \sum_{k=0}^{\infty}\frac{\delta^{(k)}}{k!}$ and $e^{-\frac{d}{dx}}\delta = \sum_{k=0}^{\infty}(-1)^k\frac{\delta^{(k)}}{k!}$ are distributions.*

Proof. Let $\psi \in C_0^{\infty}(\mathbb{R},\mathbb{R})$, then

$$\begin{aligned} \int_{\mathbb{R}} e^{\frac{d}{dx}}\delta(x)\,\psi(x)\,dx &= \int_{\mathbb{R}}\sum_{k=0}^{\infty}\frac{\delta^{(k)}}{k!}\psi(x)\,dx \\ &= \sum_{k=0}^{\infty}\int_{\mathbb{R}}\frac{\delta^{(k)}(x)}{k!}\psi(x)\,dx \\ &= \sum_{k=0}^{\infty}\int_{\mathbb{R}}\delta(x)(-1)^{(k)}\frac{\psi^{(k)}(x)}{k!}dx \\ &= \sum_{k=0}^{\infty}(-1)^{(k)}\frac{\psi^{(k)}(0)}{k!} \end{aligned}$$

and in a similar argument, one can show

$$\int_{\mathbb{R}} e^{-\frac{d}{dx}}\delta(x)\,\psi(x)\,dx = \sum_{k=0}^{\infty}\frac{\psi^{(k)}(0)}{k!}$$

with both sums $\sum_{k=0}^{\infty}\frac{\psi^{(k)}(0)}{k!}$ and $\sum_{k=0}^{\infty}(-1)^{(k)}\frac{\psi^{(k)}(0)}{k!}$ convergent to the values $e^{\frac{d}{dx}}\psi(0)$ and $e^{-\frac{d}{dx}}\psi(0)$ respectively.

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Let α be a multi index with $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ and

$$D^{\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}\cdots$$

be a partial differential operator. Let $\psi \in C_0^{\infty}(\mathbb{R}^n,\mathbb{R})$ be a function whose compact support contains zero, then we can write the Taylor series of ψ at 0 as

$$\psi(x) \backsim \sum_{|\alpha|=0}^{\infty}\frac{D^{\alpha}\psi(0)\,x^{\alpha}}{\alpha!}$$

for all x near 0 and by taking $x \to 1 = (1, 1, ...)$ of $\mathbb{R}^n$ we get

$$\sum_{|\alpha|=0}^{\infty} \frac{D^{\alpha}\psi(0)\, x^{\alpha}}{\alpha!} \underset{x\to 1=(1,1,\ldots)}{\sim} \sum_{|\alpha|=0}^{\infty} \frac{D^{\alpha}\psi(0)}{\alpha!} \backsim \psi(1)$$

Proposition 4 *$\sum_{|\alpha|=0}^{\infty} \frac{D^{\alpha}\delta}{\alpha!}$ and $\sum_{|\alpha|=0}^{\infty} (-1)^{\alpha} \frac{D^{\alpha}\delta}{\alpha!}$ are distributions.*

Proof. Let $\psi \in C_0^{\infty}(\mathbb{R}^n, \mathbb{R})$, then

$$\begin{aligned} \int_{\mathbb{R}^n} \sum_{|\alpha|=0}^{\infty} \frac{D^{\alpha}\delta(x)}{\alpha!}\psi(x)\, dx &= \int_{\mathbb{R}^n} \delta(x) \sum_{|\alpha|=0}^{\infty} (-1)^{\alpha} \frac{D^{\alpha}\psi(x)}{\alpha!} dx \\ &= \sum_{|\alpha|=0}^{\infty} (-1)^{\alpha} \frac{D^{\alpha}\psi(0)}{\alpha!} \end{aligned}$$

Similarly one can show that

$$\int_{\mathbb{R}^n} \sum_{|\alpha|=0}^{\infty} (-1)^{\alpha} \frac{D^{\alpha}\delta(x)}{\alpha!}\psi(x)\, dx = \sum_{|\alpha|=0}^{\infty} \frac{D^{\alpha}\psi(0)}{\alpha!}$$

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Proposition 5 *$e^{D^{\alpha}}\delta$ and $e^{-D^{\alpha}}\delta$ are distributions with*

$$\int_{\mathbb{R}^n} e^{D^{\alpha}}\delta(x)\,\psi(x)\, dx = e^{-D^{\alpha}}\psi(0)$$

and

$$\int_{\mathbb{R}^n} e^{-D^{\alpha}}\delta(x)\,\psi(x)\, dx = e^{D^{\alpha}}\psi(0), \forall \psi \in C_0^{\infty}(\mathbb{R}^n)$$

Proof. The proof follow from the fact that

$$e^{D^{\alpha}} = \sum_{k=0}^{\infty} \frac{D^{k\alpha}}{k!}$$

and

$$e^{-D^{\alpha}} = \sum_{k=0}^{\infty} (-1)^k \frac{D^{k\alpha}}{k!}$$

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Proposition 6 *Let $F : C_0^{\infty}(\mathbb{R}^n) \to \mathbb{R}$ be a distribution. Then $D^{\alpha}F$, $e^{D}F$ and $e^{D^{\alpha}}F$ are all distributions from $C_0^{\infty}(\mathbb{R}^n)$ to $\mathbb{R}$*

Proof. The proofs follow in a similar argument made for the Dirac delta distribution δ. ■