A Proposition( as a Corollary) to the Division Algorithm

By

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This is a result that provides the unique quotient and postive remainder, when a negative integer is divided by a positive integer whose existence are guaranteed by the actual division algorithm.

Let $n$ be an integer and $d$ be a positive integer. The division algorithm guarantees the existence of unique integers $q$, $r$ such $0 \leq r < d \leq$ with $n = qd+r$ .

The procedure first is to make an argument for a case when $n$ is positive and then we use the results obtained here to work for the case when $n$ is negative from the first case.

**Proposition**: Let $n$ be a negative integer. There are unique positive integers $q$ and $r$ such that

$$n = (-(q+1))\,d + (d-r)$$

with

$$0 \leq d-r < d.$$

**Proof.** Let $n$ be a negative integer. Then $-n$ is a positive integer and therefore from the division algorithm

$$\exists! q, r \in \mathbb{Z}$$

such that

$$-n = qd + r \quad \text{with} \quad 0 \leq r < d.$$

Putting differently :

$$\frac{-n}{d} = q + \frac{r}{d}$$

But then

$$\frac{n}{d} = -\left(\frac{-n}{d}\right) = -\left(q + \frac{r}{d}\right) = -q - \frac{r}{d}$$

Here one can see that $\frac{-r}{d}$ is a negative rational number whose absolute value is less than one. we therefore perform the following:

$$\begin{aligned} -q - \frac{r}{d} &= -q - 1 + 1 - \frac{r}{d} \\ &= -q - 1 + \frac{d-r}{d} \\ &= -(q+1) + \frac{d-r}{d} \end{aligned}$$

We now have a remainder $\frac{d-r}{d}$ which satisfies the division algorithm.

The quotient in this case is $-(q+1)$ .

$$\therefore n = (-(q+1))\, d + (d-r)$$

where $q$ and $r$ are respectively the quotient and remainder for the positive integer.