On 1-D and 2-D Poincaré's Inequality

By

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I will state and prove two propositions which are special cases on Poincaré's inequality. The first one is in $1-D$ symmetric domain where the function is odd, and the second one is in $2-D$ symmetric domain with a particular function: the exponential function of two variables.

Proposition 1 *Let $\Omega = [-a, a]$ and f be differentiable on Ω. If f is an odd function, then for $1 < p < \infty$, we have*

$$\int_{\Omega} | f |^p \, dx \leq k \int_{\Omega} | f' |^p \, dx.$$

Proof. From Poincare's inequality, and for the given index $p \in (1, \infty)$ we have :

$$\int_{\Omega} | f - f_{ave}(\Omega) |^p \leq c \int_{\Omega} | f' |^p \, dx$$

where c is some constant that depends on p and Ω. But from the oddness of f, and the symmetry of the domain Ω, we see that $f_{ave}(\Omega) = \frac{1}{\mu(\Omega)} \int_{\Omega} f d\mu = 0$ and therefore we have the desired result. ■

Proposition 2 *Let $\Omega = (-a, a)$ for $a > 0$. Prove that for $1 < p < \infty$,*

$$\int_{\Omega x \Omega} | \exp(x+y) - \frac{\sinh^2 a}{a^2} |^p \, dxdy \leq c \left(2^{\frac{p+4}{2}}\right) \left(\frac{\sinh^2 ap}{p^2}\right)$$

where c is some constant.

Proof. Again using the Poincare's inequality for $2-D$, over the domain $\Delta = \Omega \times \Omega$ we have

$$\int_{\Delta} | f - f_{ave}(\Delta) |^p \, d\mu \leq c \int_{\Delta} | Df |^p \, d\mu$$

where $d\mu$ is a surface measure in $\mathbb{R}^2$ and Df is the gradient of f . In our particular case, the function is $f(x, y) = \exp(x + y)$. Therefore the absolute value of the gradient of f is given as

$$| Df | = \sqrt{2} f(x, y)$$

Therefore,

$$| Df |^p = \left(\sqrt{2}\right)^p f^p(x,y) = \left(\sqrt{2}\right)^p f(px,py)$$

and hence

$$\begin{aligned} \int\limits_{\Delta} \quad | \quad Df |^p \, d\mu &= \int\limits_{\Omega\times\Omega} \left(\sqrt{2}\right)^p f(px,py)\, dxdy \\ &= \frac{2^{\frac{p+4}{2}}}{p^2} \sinh^2 ap \end{aligned}$$

But the average value of the exponential function f over the set $\Delta = \Omega \times \Omega$ is calculated to be :

$$\begin{aligned} f_{ave}(\Delta) &= \frac{1}{| \Omega \times \Omega |} \int\limits_{\Omega\times\Omega} f(x,y)\, dxdy \\ &= \frac{1}{4a^2} \int\limits_{\Omega\times\Omega} \exp(x+y) dxdy = \left(\frac{\sinh a}{a}\right)^2 \end{aligned}$$

Then combining all the results, the Poincare's inequality provides the following:

$$\begin{aligned} \int\limits_{\Omega\times\Omega} \quad | \quad \exp(x+y) - \left(\frac{\sinh a}{a}\right)^2 |^p \, d\mu &\leq c \int\limits_{\Delta} | Df |^p \, d\mu \\ &= c \int\limits_{\Omega\times\Omega} \left(\sqrt{2}\right)^p \exp(px+py)\, dxdy \\ &= c\frac{2^{\frac{p+4}{2}}}{p^2} \sinh^2 ap \end{aligned}$$

where c is some constant that depends on a and p and $p \in (1,\infty)$. ■