NORM ESTIMATES FOR SOLUTIONS OF ELLIPTIC BVPS OF THE DIRAC OPERATOR

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Abstract. We present norm estimates for solutions of first and second order elliptic BVPs of the Dirac operator $D = \sum_{j=1}^{n} e_j \partial_{x_j}$ considered over bounded and smooth domain Ω of $\mathbb{R}^n$. The solutions whose norms to be estimated are in some Sobolev spaces $W^{k,p}(\Omega)$ and the boundary conditions as traces of solutions and their derivatives are in some Slobodeckij spaces $W^{\lambda,p}(\partial\Omega)$ where λ is some non integer but fractional number, for $1 \leq p < \infty$ and $k \in \mathbb{Z}$.

1. Algebraic and Analytic Rudiments of Cl_n

Let $\{e_j : j = 1, 2, ..., n\}$ be an orthonormal basis for $\mathbb{R}^n$ that is equipped with an inner product so that

$$e_i e_j + e_j e_i = -2\delta_{ij} e_0 \tag{1.1}$$

where δ_{ij} is the Kronecker delta. The inner product satisfies an anti commutative relation

$$x^2 = -\|x\|^2 \tag{1.2}$$

Therefore $\mathbb{R}^n$ with these properties of base vectors generates a non commutative algebra called Clifford algebra denoted by Cl_n.

The basis of Cl_n will then be

$$\{e_A : A \subset \{1 < 2 < 3 < ... < n\}\}$$

which implies:

$$\dim(Cl_n) = 2^n$$

The object e_0 used above is the identity element of the Clifford algebra Cl_n.

Representation of elemnets of Cl_n: every $a \in Cl_n$ is represented by

$$a = \sum e_A a_A \tag{1.3}$$

where a_A is a real number.

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Thus every $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ can be identified with $\sum_{j=1}^{n} e_j x_j$ of Cl_n and therefore we have an embedding

$$\mathbb{R}^n \hookrightarrow Cl_n$$

We also define what is called a Clifford conjugate of

$$a = \sum e_A a_A$$

as

$$\overline{a} = \sum \overline{e}_A a_A$$

where

$$\overline{e_{j_1}...e_{j_r}} = (-1)^r \, e_{j_r}...e_{j_1}$$

For instance for $i, j = 1, 2, ..., n$,

$$\overline{e}_j = -e_j, \quad e_j^2 = -1$$

and for

$$i \neq j : \overline{e_i e_j} = (-1)^2 e_j e_i = e_j e_i$$

Definition 1. *We define the Clifford norm of*

$$a = \sum e_A a_A \in Cl_n$$

by

$$\|a\| = \left((a\overline{a})_0\right)^{\frac{1}{2}} = \left(\sum_A a_A^2\right)^{\frac{1}{2}} \tag{1.4}$$

where $(a)_0$ is the real part of $a\overline{a}$.

The norm $\|.\|$ satisfies the inequality:

$$\|ab\| \leq c\,(n)\, \|a\| \|b\| \tag{1.5}$$

with $c\,(n)$ a dimensional constant.

Also each non zero element $x \in \mathbb{R}^n$ has an inverse given by :

$$x^{-1} = \frac{\overline{x}}{\|x\|^2} \tag{1.6}$$

$\lessdot$ In the article it is always the case that $1 < p < \infty$ unless otherwise specified and Ω is a bounded and smooth (at least with C^1 - boundary $\partial\Omega$) domain of $\mathbb{R}^n$

A Clifford valued (Cl_n- valued) function f defined on Ω as

$$f : \Omega \longrightarrow Cl_n$$

has a representation

$$f = \sum_A e_A f_A \tag{1.7}$$

where $f_A : \Omega \longrightarrow \mathbb{R}$ is a real valued component or section of f.

Definition 2. *For a function $f \in C^1(\Omega) \cap C\left(\overline{\Omega}\right)$, we define the Dirac derivative of f by*

$$Df(x) = \sum_{j=1}^{n} e_j \partial_{x_j} f(x) \tag{1.8}$$

A function $f : \Omega \longrightarrow Cl_n$ is called left monogenic or left Clifford analytic over Ω if

$$Df(x) = 0, \ \forall x \in \Omega$$

and likewise it is called right monogenic over Ω if

$$f(x)D = \sum_{j=1}^{n} \partial_{x_j} f(x)\, e_j = 0, \ \forall x \in \Omega$$

An example of both left and right monogenic function defined over $\mathbb{R}^n \backslash \{0\}$ is given by

$$\psi(x) = \frac{\overline{x}}{\omega_n \|x\|^n}$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

The function ψ is also a fundamental solution to the Dirac operator D and we define integral transforms as convolutions of ψ with functions of some function spaces below.

Definition 3. *Let $f \in C^1(\Omega, Cl_n) \cap C\left(\overline{\Omega}\right)$.*
We define two integral transforms as follow:

$$\zeta_\Omega f(x) = \int_\Omega \psi(y - x) f(y)\, d\Omega_y, \quad x \in \Omega \tag{1.9}$$

$$\xi_{\partial\Omega} f(x) = \int_{\partial\Omega} \psi(y - x)\, \upsilon(y) f(y)\, d\partial\Omega_y, \quad x \notin \partial\Omega \tag{1.10}$$

The integral transform defined in (1.9) a domain integral is called the Theodorescu transform or the Cauchy transform. It is a convolution $\psi * f$ over Ω. The integral transform defined in (1.10) is some times called the *Feuter* transform as a boundary integral which again is a convolution $\psi * \upsilon f$ over $\partial\Omega$. $\upsilon(y)$ is a unit normal vector pointing outward at $y \in \partial\Omega$.

2. Sobolev and Slobodeckij Spaces

Definition 4. *For $1 < p < \infty$, $k \in \mathbb{N} \cup \{0\}$ we define:*

I: The Sobolev space $W^{k,p}(\Omega)$ as

$$W^{k,p}(\Omega) := \{f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega), \quad \|\alpha\| \leq k\}$$

with norm

$$\|f\|_{W^{k,p}(\Omega)} = \left(\sum_{\|\alpha\| \le k} \int_{\Omega} |D^{\alpha} f|^{p} dx \right)^{\frac{1}{p}} \tag{2.1}$$

II: The Slobodeckij spaces for $0 < \lambda < 1$ as

$$W^{\lambda,p}(\partial\Omega) := \{ f \in L^{p}(\partial\Omega) : \int_{\partial\Omega} \int_{\partial\Omega} \frac{|f(x) - f(y)|^{p}}{|x-y|^{n+\lambda p-1}} d\sigma_{x} d_{\sigma y} < \infty \}$$

and norm is defined by

$$\|f\|_{W^{\lambda,p}(\partial\Omega)} = \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|f(x) - f(y)|^{p}}{|x-y|^{n+\lambda p-1}} d\sigma_{x} d_{\sigma y} \right)^{\frac{1}{p}} \tag{2.2}$$

III: The Slobodeckij spaces for $\lambda = [\lambda] + \{\lambda\}$ where $0 < \{\lambda\} < 1$:

$$W^{\lambda,p}(\partial\Omega) := \{ f \in W^{[\lambda],p}(\partial\Omega) : \sum_{\|\alpha\| \le [\lambda]} \int_{\partial\Omega} |Df|^{p} d\sigma_{x} + \sum_{\|\alpha\| = [\lambda]} \int_{\partial\Omega} \int_{\partial\Omega} \frac{|D^{\alpha} f(x) - D^{\alpha} f(y)|^{p}}{|x-y|^{n+\{\lambda\}p-1}} d\sigma_{x} d_{\sigma y} < \infty \}$$

and hence norm is given by

$$\|f\|_{W^{\lambda,p}(\partial\Omega)} = \left(\sum_{\|\alpha\| \le [\lambda]} \int_{\partial\Omega} |Df|^{p} d\sigma_{x} + \sum_{\|\alpha\| = [\lambda]} \int_{\partial\Omega} \int_{\partial\Omega} \frac{|D^{\alpha} f(x) - D^{\alpha} f(y)|^{p}}{|x-y|^{n+\{\lambda\}p-1}} d\sigma_{x} d_{\sigma y} \right)^{\frac{1}{p}} \tag{2.3}$$

In the definitions of the Slobodeckij spaces and associated norms, the irregularity exponent $n + \{\lambda\}p - 1$ is due to the fact that the dimension of $\partial\Omega$ is $n-1$ and $d\sigma$ is a hypersurface measure on $\partial\Omega$.

Slobodeckij spaces as subspaces of Sobolev spaces but with fractional exponents are analogues of the Hölder spaces in classical spaces of continuous functions.

3. Some Properties and Relations Between D, ζ_{Ω}, τ and $\xi_{\partial\Omega}$

Proposition 1. *$D : W^{k,p}(\Omega, Cl_n) \longrightarrow W^{k-1,p}(\Omega, Cl_n)$ is continuous with*

$$\|Df\|_{W^{k-1,p}(\Omega, Cl_n)} \le \gamma \|f\|_{W^{k,p}(\Omega, Cl_n)}$$

for $\gamma = \gamma(n, p, \Omega)$ a positive constant.

Proof. Let $f \in W^{k,p}(\Omega, Cl_n)$. We need to show that

$$\|Df\|_{W^{k-1,p}(\Omega, Cl_n)} \le c \|f\|_{W^{k,p}(\Omega, Cl_n)}$$

for some positive constant c.

$$f \in W^{k,p}(\Omega, Cl_n) \Longrightarrow \|f\|_{W^{k,p}(\Omega, Cl_n)} = \left(\sum_{\|\alpha\| \le k} \int_{\Omega} |D^{\alpha} f|^{p} dx \right)^{\frac{1}{p}} < \infty$$

But then

$$\begin{aligned}
\|Df\|_{W^{k-1,p}(\Omega,Cl_n)} &= \left(\sum_{\|\alpha\|\le k-1} \int_\Omega |D^\alpha f|^p dx \right)^{\frac{1}{p}} \\
&\le \left(\sum_{\|\alpha\|\le k-1} \int_\Omega |D^\alpha f|^p dx + \sum_{\|\alpha\|=k-1} \int_\Omega |D^\alpha f|^p dx \right)^{\frac{1}{p}} \\
&= \left(\sum_{\|\alpha\|\le k} \int_\Omega |D^\alpha f|^p dx \right)^{\frac{1}{p}} \\
&= \|f\|_{W^{k,p}(\Omega,Cl_n)}
\end{aligned}$$

Therefore for $c = 1$, the proposition is proved. □

Proposition 2. *$D : L^p(\Omega) \longrightarrow W^{-1,p}(\Omega)$ is continuous for $1 < p < \infty$.*

Proof. Let $f \in L^p(\Omega)$. Then

$$\|Df\|_{W^{-1,p}(\Omega)} = \sup\{\frac{|\langle Df, v\rangle|}{\|v\|_{W_0^{1,q}(\Omega)}} : v \ne 0, v \in W_0^{1,q}(\Omega)\}$$

for $p^{-1} + q^{-1} = 1$.

But

$$|\langle Df, v\rangle| = |\langle f, Dv\rangle| \le \|f\|_{L^p(\Omega)} \|Dv\|_{L^q(\Omega)} \le \|f\|_{L^p(\Omega)} \|v\|_{W_0^{1,q}(\Omega)}$$

Thus by the Cauchy-Schwartz inequality we have

$$\frac{|\langle Df, v\rangle|}{\|v\|_{W_0^{1,q}(\Omega)}} \le \frac{\|f\|_{L^p(\Omega)} \|v\|_{W_0^{1,q}(\Omega)}}{\|v\|_{W_0^{1,q}(\Omega)}} = \|f\|_{L^p(\Omega)}$$

Therefore

$$\begin{aligned}
\|Df\|_{W^{-1,p}(\Omega)} &= \sup\{\frac{|\langle Df, v\rangle|}{\|v\|_{W_0^{1,q}(\Omega)}} : v \ne 0,\ v \in W_0^{1,q}(\Omega)\} \\
&\le \sup\{\frac{\|f\|_{L^p(\Omega)} \|v\|_{W_0^{1,q}(\Omega)}}{\|v\|_{W_0^{1,q}(\Omega)}} : v \ne 0,\ v \in W_0^{1,q}(\Omega)\} \\
&= \|f\|_{L^p(\Omega)}
\end{aligned}$$

□

Proposition 3. *(Mapping properties) ([4], [6])*

Let $k \in \mathbb{N} \cup \{0\}$ *and* $1 < p < \infty$. *Then there are positive constants* $\beta = \beta\,(n, p, \Omega)$, $\theta = \theta\,(n, p, \Omega)$ *and* $\delta = \delta\,(n, p, \Omega)$ *such that*

$$\zeta_\Omega : W^{k,p}\,(\Omega, Cl_n) \longrightarrow W^{k+1,p}\,(\Omega, Cl_n) \tag{3.1}$$

with

$$\|\zeta_\Omega f\|_{W^{k+1,p}(\Omega,Cl_n)} \leq \beta \|f\|_{W^{k,p}(\Omega,Cl_n)}$$

$$\xi_{\partial\Omega} : W^{\lambda,p}\,(\partial\Omega, Cl_n) \longrightarrow W^{\lambda+\frac{1}{p},p}\,(\Omega, Cl_n) \tag{3.2}$$

with

$$\|\xi_{\partial\Omega} f\|_{W^{\lambda+\frac{1}{p},p}(\Omega,Cl_n)} \leq \theta \|f\|_{W^{\lambda,p}(\partial\Omega,Cl_n)}$$

and

$$\tau : W^{k,p}(\Omega, Cl_n) \longrightarrow W^{k-\frac{1}{p},p}\,(\partial\Omega, Cl_n) \tag{3.3}$$

is the trace operator with

$$\sum_{\|\alpha\| \leq [\lambda+\frac{1}{p}]} \int_\Omega |D^\alpha \tau f|^p dx + \sum_{\|\alpha\| = [\lambda+\frac{1}{p}]} \int_\Omega \int_\Omega \frac{|D^\alpha \tau f(x) - D^\alpha \tau f(y)|^p}{|x-y|^{n+\{\lambda+\frac{1}{p}\}p}} dxdy$$

$$\leq \quad \delta^p \left(\sum_{\|\alpha\| \leq [\lambda]} \int_{\partial\Omega} |D^\alpha f|^p dx + \sum_{\|\alpha\| = [\lambda]} \int_{\partial\Omega} \int_{\partial\Omega} \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{n-1+\{\lambda+\frac{1}{p}\}p}} d\sigma_x d\sigma_y \right)$$

Proposition 4. *The composition* $\xi_{\partial\Omega} \circ \tau$ *preserves regularity of a function in a Sobolev space.*

Proof. Indeed, τ makes a function to loose a regularity fractional exponent of $\frac{1}{p}$ when taken along the boundary of the domain. But the boundary or *Feuter* integral $\xi_{\partial\Omega}$ augments the regularity exponent of a function defined on the boundary by an exponent of $\frac{1}{p}$.

Therefore the composition operator $\xi_{\partial\Omega} \circ \tau$ preserves or fixes the regularity exponent of a function in a Sobolev space.

□

Proposition 5. *(Borel-Pompeiu)*
Let $f \in W^{k,p}\,(\Omega, Cl_n)$. *Then*

$$f = \xi_{\partial\Omega} \tau f + \zeta_\Omega D f$$

Corollary 1. *(i) If* $f \in W_0^{k,p}\,(\Omega, Cl_n)$, *then*

$$f = \zeta_\Omega D f$$

That is D *is a right inverse for* ζ_Ω *and* ζ_Ω *is a left inverse for* D *over traceless spaces.*

(ii) If f is monogenic function over Ω, then

$$f = \xi_{\partial\Omega}\tau f$$

Therefore monogenic functions are always Cauchy transforms of their traces over the boundary.

4. Elliptic First and Second Order BVPs

Proposition 6. *Let $f \in W^{k-1,p}(\Omega, Cl_n)$ for $k \geq 1$. Then the first order elliptic BVP:*

$$\begin{cases} Du = f & in\ \Omega \\ \tau u = g & on\ \partial\Omega \end{cases} \tag{4.1}$$

has a solution $u \in W^{k,p}(\Omega, Cl_n)$ given by

$$u(x) = \xi_{\partial\Omega} g + \zeta_{\Omega} f$$

Proof. The proof follows from the Borel-Pompeiu relation. As to where exactly u and g belong, we make the argument : f is in $W^{k-1,p}(\Omega, Cl_n)$ and hence from the mapping property of D, we have u to be a function in $W^{k,p}(\Omega, Cl_n)$.

Also from the mapping property of the trace operator τ we have

$$\tau u = u|_{\partial\Omega} = g \in W^{k-\frac{1}{p},p}(\partial\Omega, Cl_n)$$

□

Proposition 7. *The solution $u \in W^{k,p}(\Omega, Cl_n)$ has a norm estimate :*

$$\begin{aligned} \|u\|_{W^{k,p}(\Omega,Cl_n)} \leq\ & \gamma_1 \left(\sum_{\|\alpha\| \leq k-1} \int_{\partial\Omega} |D^\alpha g|^p d\sigma x + \sum_{\|\alpha\|=k-1} \int_{\partial\Omega}\int_{\partial\Omega} \frac{|D^\alpha g(x) - D^\alpha g(y)|^p}{|x-y|^{n+p-2}} d\sigma_x d\sigma_y \right)^{\frac{1}{p}} \\ & + \gamma_2 \left(\sum_{\|\alpha\|=k-1} \int_{\partial\Omega} |f|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

where γ_1, γ_2 are constants the depend on p,n and Ω.

Proof. First let us determine regularity exponents of

$$g \in W^{k-\frac{1}{p},p}(\partial\Omega, Cl_n)$$

For the regularity index $k - \frac{1}{p}$ the integer part is

$$[k - \frac{1}{p}] = k - 1$$

and the fractional part is

$$\{k-\frac{1}{p}\}=1-\frac{1}{p}$$

Besides $\dim(\partial\Omega)=n-1$. From the mapping properties of D, ζ_Ω, τ and $\xi_{\partial\Omega}$, we have

$$u\in W^{k,p}(\Omega,Cl_n)$$

and

$$\tau u=g\in W^{k-\frac{1}{p},p}(\partial\Omega,Cl_n)$$

Therefore the solution u given by:

$$u(x)=\xi_{\partial\Omega}g+\zeta_\Omega f$$

has norm estimate

$$\begin{aligned}
\|u\|_{W^{k,p}(\Omega,Cl_n)} &= \|\xi_{\partial\Omega}g+\zeta_\Omega f\|_{W^{k,p}(\Omega,Cl_n)}\\
&\leq \|\xi_{\partial\Omega}g\|_{W^{k,p}(\Omega,Cl_n)}+\|\zeta_\Omega f\|_{W^{k,p}(\Omega,Cl_n)}\\
&\leq \gamma_1\|g\|_{W^{k-\frac{1}{p},p}(\partial\Omega,Cl_n)}+\gamma_2\|f\|_{W^{k-1,p}(\Omega,Cl_n)}\\
&= \gamma_1\left(\sum_{\|\alpha\|\leq k-1}\int_{\partial\Omega}|D^\alpha g|^p d\sigma x+\sum_{\|\alpha\|=k-1}\int_{\partial\Omega}\int_{\partial\Omega}\frac{|D^\alpha g(x)-D^\alpha g(y)|^p}{|x-y|^{n-1+\{k-\frac{1}{p}\}p}}d\sigma_x d\sigma_y\right)^{\frac{1}{p}}\\
&\quad+\gamma_2\left(\sum_{\|\alpha\|=k-1}\int_{\partial\Omega}|f|^p dx\right)^{\frac{1}{p}}\\
&= \gamma_1\left(\sum_{\|\alpha\|\leq k-1}\int_{\partial\Omega}|D^\alpha g|^p d\sigma x+\sum_{\|\alpha\|=k-1}\int_{\partial\Omega}\int_{\partial\Omega}\frac{|D^\alpha g(x)-D^\alpha g(y)|^p}{|x-y|^{n+p-2}}d\sigma_x d\sigma_y\right)^{\frac{1}{p}}\\
&\quad+\gamma_2\left(\sum_{\|\alpha\|=k-1}\int_{\partial\Omega}|f|^p dx\right)^{\frac{1}{p}}
\end{aligned}$$

The constants γ_1 and γ_2 are from the mapping properties of $\xi_{\partial\Omega},\zeta_\Omega$ and τ. □

Proposition 8. *Let $f\in W^{k,p}(\Omega,Cl_n)$. Then the second order elliptic BVP*

$$\begin{cases} -\Delta u=f & \text{in } \Omega\\ \tau Du=g_1 & \text{on } \partial\Omega\\ \tau u=g_2 & \text{on } \partial\Omega\end{cases}\tag{4.2}$$

has a solution given by

$$u=\xi_{\partial\Omega}(g_2)+\zeta_\Omega\xi_{\partial\Omega}(g_1)+\zeta_\Omega\circ\zeta_\Omega(f)$$

in $W^{k+2,p}(\Omega)$ with

$$g_1\in W^{k+1-\frac{1}{p},p}(\partial\Omega),\quad g_2\in W^{k+2-\frac{1}{p},p}(\partial\Omega)$$

Proof. As $f \in W^{k,p}(\Omega, Cl_n)$, the solution u is in the Sobolev space $W^{k+2,p}(\Omega)$ and hence

$$\tau u = g_2 \in W^{k+2-\frac{1}{p},p}(\partial\Omega)$$

But then Du is in $W^{k+1,p}(\Omega)$ and hence

$$\tau Du = g_1$$

is in the Slobodeckij space $W^{k+1-\frac{1}{p},p}(\partial\Omega)$.

The solution u of the BVP is obtained by repeated application of the Borel-Pompeiu formula by writing the Laplacian Δ as $-D^2$.

Now let us first determine the integer and fractional parts of indices $k+2-\frac{1}{p}$ and $k+1-\frac{1}{p}$ as

$$\begin{aligned} [k+2-\frac{1}{p}] &= k+1, \quad \{k+2-\frac{1}{p}\} = 1-\frac{1}{p} \\ [k+1-\frac{1}{p}] &= k, \quad \{k+1-\frac{1}{p}\} = 1-\frac{1}{p} \end{aligned}$$

Therefore from the properties of the mappings studied above, we have a norm estimate of the solution u in $W^{k+2,p}(\Omega)$ in terms of norms of f, g_1 and g_2 as follow:

$$\begin{aligned} \|u\|_{W^{k+2,p}(\Omega)} &= \|\xi_{\partial\Omega}(g_2) + \zeta_\Omega \xi_{\partial\Omega}(g_1) + \zeta_\Omega \circ \zeta_\Omega (f)\|_{W^{k+2,p}(\Omega)} \\ &\leq \gamma_1 \left(\sum_{\|\alpha\| \leq k+1} \int_{\partial\Omega} |D^\alpha g_2|^p d\sigma_x + \sum_{\|\alpha\|=k+1} \int_{\partial\Omega}\int_{\partial\Omega} \frac{|D^\alpha g_2(x) - D^\alpha g_2(y)|^p}{|x-y|^{n+p-2}} d\sigma_x d\sigma_y \right)^{\frac{1}{p}} \\ &\quad + \gamma_2 \left(\sum_{\|\alpha\| \leq k} \int_{\partial\Omega} |D^\alpha g_1|^p d\sigma_x + \sum_{\|\alpha\|=k} \int_{\partial\Omega}\int_{\partial\Omega} \frac{|D^\alpha g_1(x) - D^\alpha g_1(y)|^p}{|x-y|^{n+p-2}} d\sigma_x d\sigma_y \right)^{\frac{1}{p}} \\ &\quad + \gamma_3 \left(\sum_{\|\alpha\| \leq k} \int_{\partial\Omega} |D^\alpha f|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

for some positive constants γ_1, γ_2 and γ_3 that depend on p, n, Ω □

Proposition 9. *For the BVP (4.1) there exist positive constants c, γ_1 and γ_2 such that the solution $u \in W^{k,2n}(\Omega)$ satisfies the norm estimate:*

$$
\begin{aligned}
& c^{-1}\left(\sup_{\substack{x,y\in\Omega\\ x\neq y}}\frac{|u\left(x\right)-u\left(y\right)|}{|x-y|^{\frac{1}{2}}}+\|u\|_{C(\Omega)}\right)\\
\leq\;& \gamma_1\left(\sum_{\|\alpha\|\leq k-1}\int_{\partial\Omega}|D^{\alpha}g|^{2n}d\sigma x+\sum_{\|\alpha\|=k-1}\int_{\partial\Omega}\int_{\partial\Omega}\frac{|D^{\alpha}g\left(x\right)-D^{\alpha}g\left(y\right)|^{2n}}{|x-y|^{n+p-2}}d\sigma_x d\sigma_y\right)^{\frac{1}{2n}}\\
& +\gamma_2\left(\sum_{\|\alpha\|=k-1}\int_{\partial\Omega}|f|^{2n}dx\right)^{\frac{1}{2n}}
\end{aligned}
$$

Proof. From the Sobolev embeding theorems, if $p > n$, then

$$W^{k,p}\left(\Omega\right)\hookrightarrow C^{0,\lambda}\left(\Omega\right)$$

for $0<\lambda\leq 1-\frac{n}{p}$.

But then for $p = 2n$, we have $0 < \lambda \leq \frac{1}{2}$ and thuerefore the solution u which is in $W^{k,2n}\left(\Omega\right)$ is contained in Hölder spaces $C^{0,\lambda}\left(\Omega\right)$.

Thus $\exists\ c = c(p,n,\Omega) > 0$ such that

$$c^{-1}\|u\|_{C^{0,\lambda}(\Omega)}\leq\|u\|_{W^{k,2n}(\Omega)}$$

That is

$$
\begin{aligned}
& c^{-1}\left(\sup_{\substack{x,y\in\Omega\\ x\neq y}}\frac{|u\left(x\right)-u\left(y\right)|}{|x-y|^{\lambda}}+\|u\|_{C(\Omega)}\right)\\
\leq\;& \|u\|_{W^{k,2n}(\Omega)}\\
\leq\;& \gamma_1\left(\sum_{\|\alpha\|\leq k-1}\int_{\partial\Omega}|D^{\alpha}g|^{2n}d\sigma x+\sum_{\|\alpha\|=k-1}\int_{\partial\Omega}\int_{\partial\Omega}\frac{|D^{\alpha}g\left(x\right)-D^{\alpha}g\left(y\right)|^{2n}}{|x-y|^{n+p-2}}d\sigma_x d\sigma_y\right)^{\frac{1}{2n}}\\
& +\gamma_2\left(\sum_{\|\alpha\|=k-1}\int_{\partial\Omega}|f|^{2n}dx\right)^{\frac{1}{2n}}
\end{aligned}
$$

Choosing $\lambda = \frac{1}{2}$, we have the required result. □

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